HW 4 Solutions

1. a) mass balance, steady-state, on cells, X

$$-QX + V\mu X = 0$$

$$\frac{1}{\tau} = \mu = \frac{\hat{\mu} S}{K_s + S}$$

rearrange : $\frac{K_s}{\tau} + \frac{S}{\tau} = \hat{\mu} S$

$$S\left(\hat{\mu} - \frac{1}{\tau}\right) = \frac{K_s}{\tau}$$

$$\boxed{S = \frac{K_s}{(\hat{\mu}\tau - 1)}}$$

mass balance on S

$$Q(S_o - S) + V r_s = 0$$

$$-r_s = \frac{S_o - S}{\tau} = \frac{\mu X}{Y}$$

$$X = \frac{Y(S_o - S)}{\mu\tau}, \quad \mu = \frac{1}{\tau}, \quad \boxed{X = Y(S_o - S)}$$

a) $\tau = \tau_{min}$, $S = S_o$

$$\frac{1}{\tau_{min}} = \frac{2d^{-1}(250)}{(50 + 250)}, \quad \boxed{\tau_{min} = 0.6 d}$$

b) for $S = 25$ mg/L

$$\tau(S) = \frac{K_s + S}{\hat{\mu} S} = \frac{50 + 25}{2(25)} d = \boxed{1.5 d}$$

$V = 10^4 m^3/d \times 1.5 d = 1.5 \times 10^4 m^3$

c) $X(1.5d = \tau) = 0.6(250 - 25) = 135 \frac{mg}{l} COD_X$

1d) for ideal PFR @ steady-state:

coupled mass balances for S and X

$$\frac{dS}{d\tau} = -\left(\frac{\hat{\mu}}{Y}\right)\frac{X\,S}{(K_s+S)} \qquad \text{where } X_o > 0$$

$$\frac{dX}{d\tau} = \hat{\mu} X \frac{S}{(K_s+S)} \qquad \text{where } X_o > 0$$

numerical solution for $5 \le X_o \le 50$

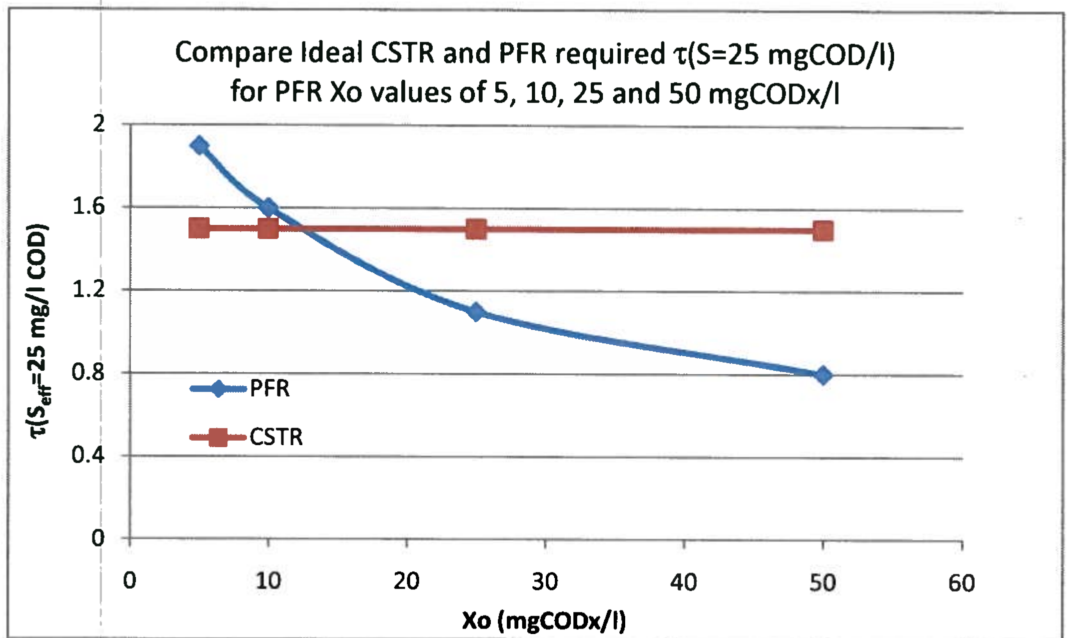
X_o (mg COD$_X$/L)	τ_{PFR} ($S = 25$ mg COD/L) (d)
5	1.9
10	1.6
25	1.1
50	0.8

(see numerical solution tables and graphs following)

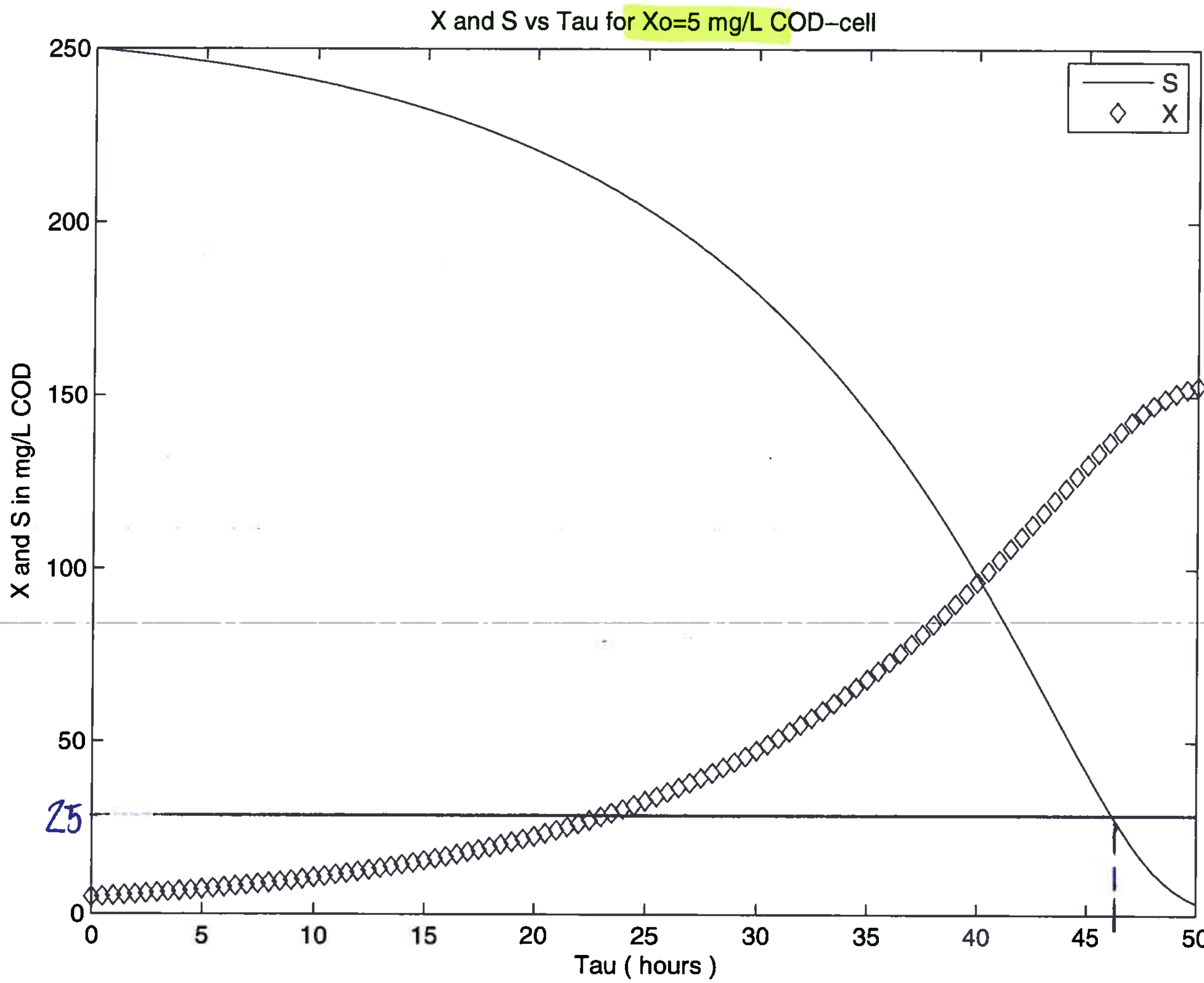
Problem 1d, comparison of ideal CSTR and PFR results from numerical solution of coupled mass balances for S and X, assuming values for Xo of 5, 10, 25, and 50 mgCODx/l for S effluent = 25 mg/l COD

Xo (mgCODx/l)	τ_{pfr} (d) PFR	τ_{cstr} (d) CSTR
5	1.9	1.5
10	1.6	1.5
25	1.1	1.5
50	0.8	1.5

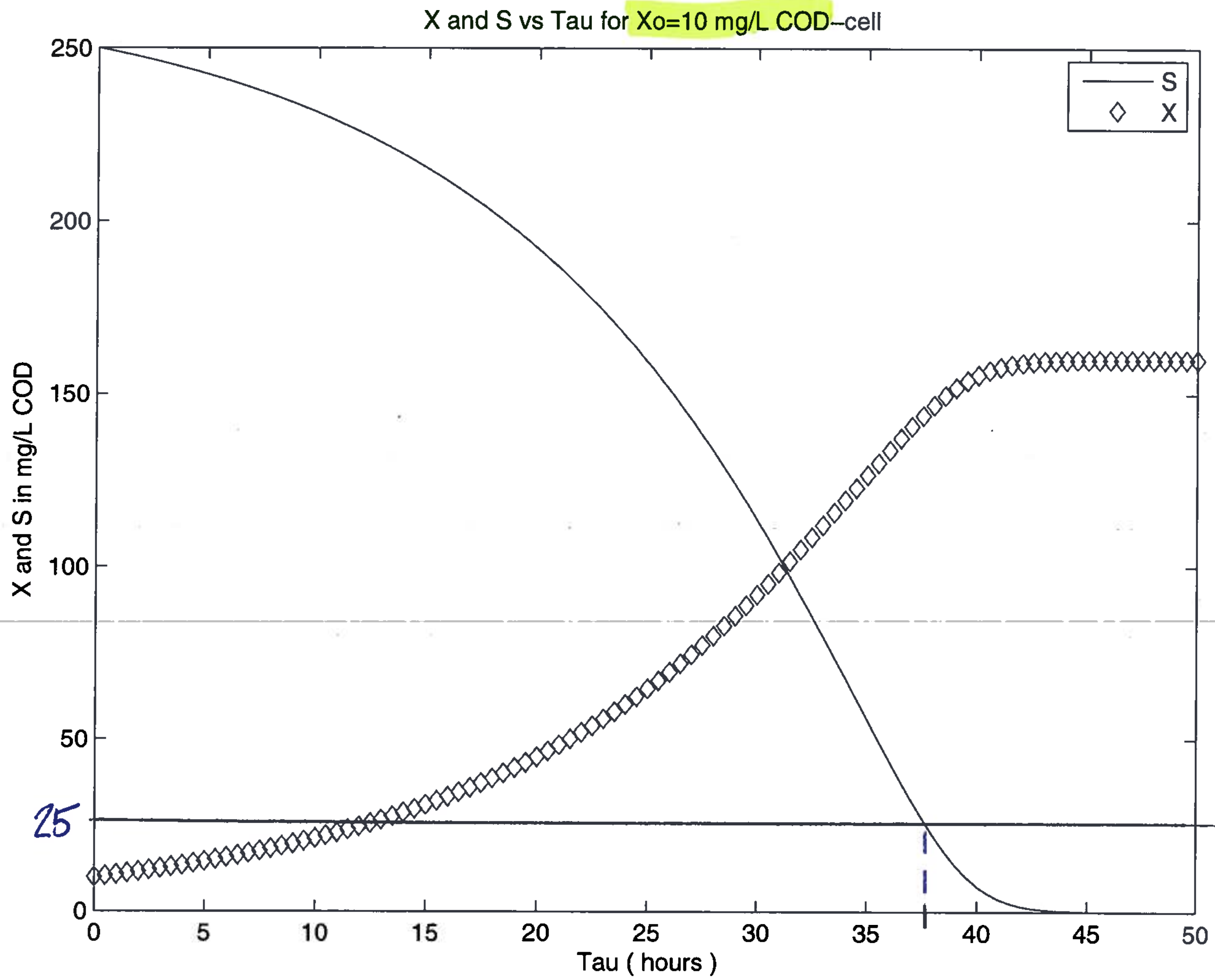
Note that for Xo = 10 mgCODx/l,
coupled mass balance solution is close
to assuming X_{avg} for PFR = 67.5 mgCODx/l



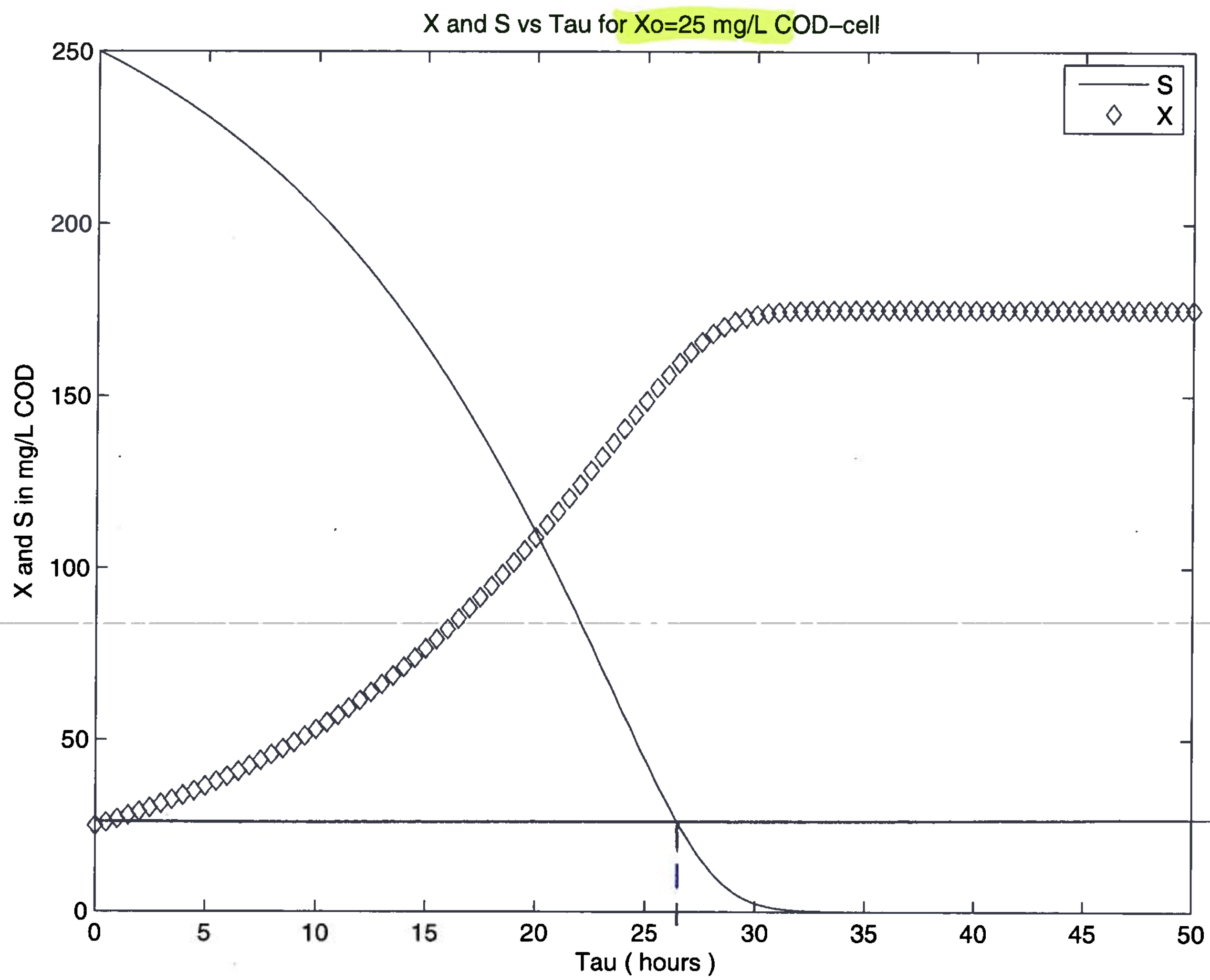
1d 4



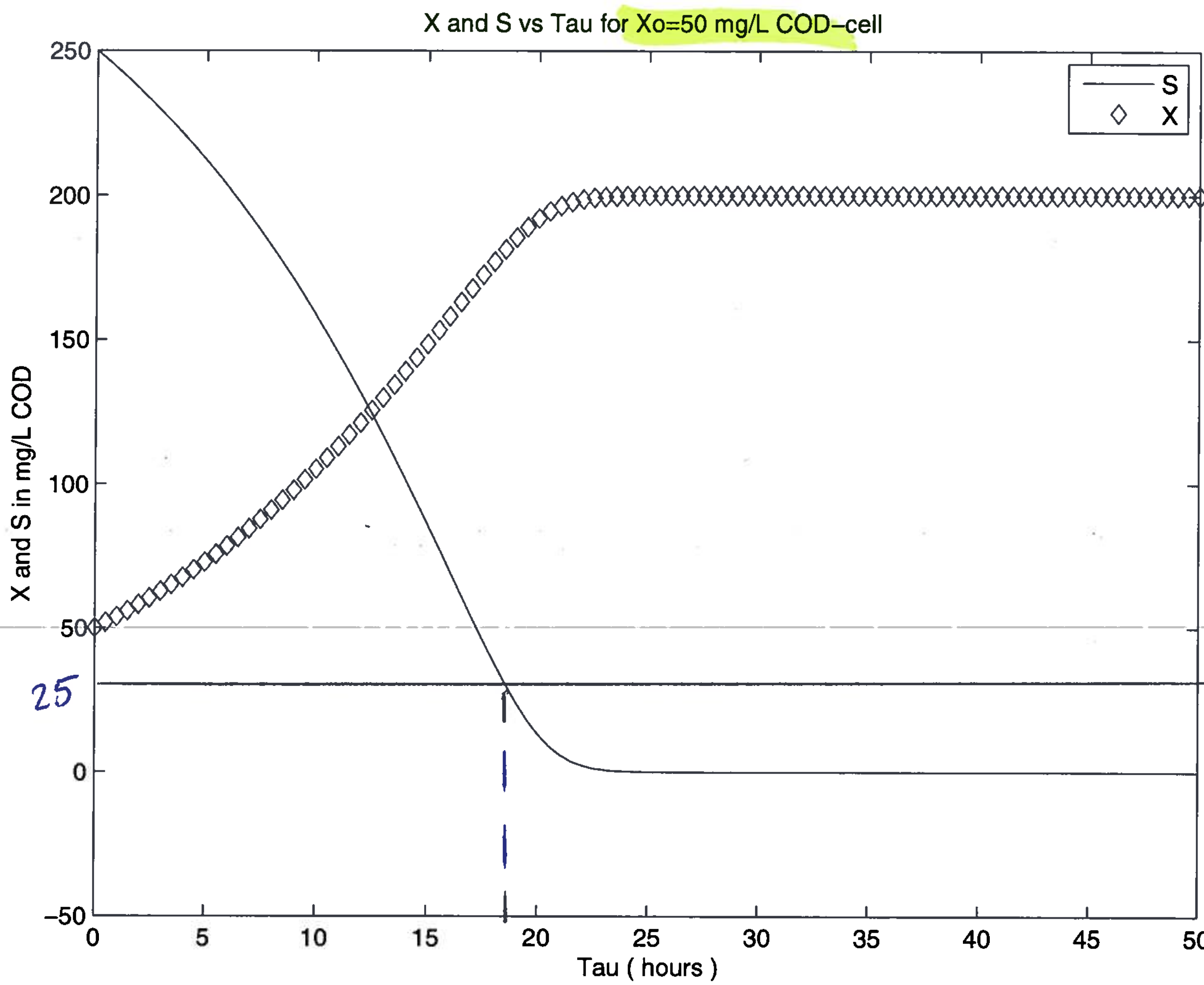
X and S vs Tau for Xo=5 mg/L COD-cell
X and S in mg/L COD
Tau (hours)
S
X
250
200
150
100
50
25
0
0
5
10
15
20
25
30
35
40
45
50


X and S vs Tau for Xo=10 mg/L COD--cell
S
X
X and S in mg/L COD
Tau (hours)
250
200
150
100
50
25
0
0
5
10
15
20
25
30
35
40
45
50

1d(6)


X and S vs Tau for Xo=25 mg/L COD–cell
S
X
X and S in mg/L COD
Tau (hours)
0
50
100
150
200
250
0
5
10
15
20
25
30
35
40
45
50

1d 7



X and S vs Tau for Xo=50 mg/L COD–cell
S
X
X and S in mg/L COD
250
200
150
100
50
25
0
−50
0
5
10
15
20
25
30
35
40
45
50
Tau (hours)

1d(8)

Time(Hrs)	X (mg/L COD)	S (mg/L COD)
0.0	10.0	250.0
0.5	10.4	249.4
1.0	10.8	248.7
1.5	11.2	248.0
2.0	11.6	247.3
2.5	12.1	246.5
3.0	12.5	245.8
3.5	13.0	244.9
4.0	13.5	244.1
4.5	14.1	243.2
5.0	14.6	242.3
5.5	15.2	241.4
6.0	15.7	240.4
6.5	16.3	239.4
7.0	17.0	238.4
7.5	17.6	237.3
8.0	18.3	236.2
8.5	19.0	235.0
9.0	19.7	233.8
9.5	20.5	232.5
10.0	21.3	231.2
10.5	22.1	229.8
11.0	22.9	228.4
11.5	23.8	227.0
12.0	24.7	225.5
12.5	25.7	223.9
13.0	26.7	222.2
13.5	27.7	220.5
14.0	28.7	218.8
14.5	29.8	217.0
15.0	31.0	215.1
15.5	32.1	213.1
16.0	33.4	211.1
16.5	34.6	209.0
17.0	35.9	206.8
17.5	37.3	204.5
18.0	38.7	202.2
18.5	40.2	199.7
19.0	41.7	197.2
19.5	43.2	194.6
20.0	44.9	191.9
20.5	46.5	189.1
21.0	48.3	186.2
21.5	50.1	183.2
22.0	52.0	180.1
22.5	53.9	176.8
23.0	55.9	173.5
23.5	58.0	170.0
24.0	60.1	166.5
24.5	62.3	162.8
25.0	64.6	159.0

Time(Hrs)	X (mg/L COD)	S (mg/L COD)
25.5	67.0	155.0
26.0	69.4	151.0
26.5	71.9	146.8
27.0	74.5	142.4
27.5	77.2	138.0
28.0	80.0	133.3
28.5	82.8	128.6
29.0	85.8	123.7
29.5	88.8	118.7
30.0	91.9	113.5
30.5	95.1	108.2
31.0	98.3	102.8
31.5	101.7	97.2
32.0	105.1	91.6
32.5	108.5	85.8
33.0	112.1	79.9
33.5	115.6	73.9
34.0	119.3	67.9
34.5	122.9	61.8
35.0	126.5	55.8
35.5	130.2	49.7
36.0	133.7	43.8
36.5	137.3	37.9
37.0	140.6	32.3
37.5	143.8	26.9
38.0	146.8	22.0
38.5	149.5	17.4
39.0	151.9	13.5
39.5	154.0	10.1
40.0	155.6	7.3
40.5	156.9	5.2
41.0	157.9	3.6
41.5	158.6	2.4
42.0	159.0	1.6
42.5	159.4	1.0
43.0	159.6	0.7
43.5	159.7	0.4
44.0	159.8	0.3
44.5	159.9	0.2
45.0	159.9	0.1
45.5	160.0	0.1
46.0	160.0	0.0
46.5	160.0	0.0
47.0	160.0	0.0
47.5	160.0	0.0
48.0	160.0	0.0
48.5	160.0	0.0
49.0	160.0	0.0
49.5	160.0	0.0
50.0	160.0	0.0

1d(9)

Time(Hrs)	X (mg/L COD)	S (mg/L COD)
0.0	25.0	250.0
0.5	26.0	248.4
1.0	27.0	246.7
1.5	28.0	245.0
2.0	29.1	243.2
2.5	30.2	241.3
3.0	31.4	239.4
3.5	32.6	237.4
4.0	33.8	235.3
4.5	35.1	233.1
5.0	36.5	230.9
5.5	37.9	228.6
6.0	39.3	226.2
6.5	40.8	223.6
7.0	42.4	221.1
7.5	44.0	218.4
8.0	45.7	215.6
8.5	47.4	212.7
9.0	49.2	209.7
9.5	51.1	206.6
10.0	53.0	203.4
10.5	55.0	200.0
11.0	57.1	196.6
11.5	59.2	193.0
12.0	61.4	189.3
12.5	63.7	185.4
13.0	66.1	181.5
13.5	68.6	177.4
14.0	71.1	173.1
14.5	73.8	168.7
15.0	76.5	164.2
15.5	79.3	159.5
16.0	82.2	154.7
16.5	85.2	149.7
17.0	88.3	144.6
17.5	91.5	139.2
18.0	94.7	133.8
18.5	98.1	128.2
19.0	101.6	122.4
19.5	105.1	116.4
20.0	108.8	110.4
20.5	112.5	104.1
21.0	116.3	97.8
21.5	120.2	91.3
22.0	124.2	84.7
22.5	128.2	77.9
23.0	132.3	71.2
23.5	136.4	64.4
24.0	140.5	57.5
24.5	144.5	50.8
25.0	148.5	44.1

Time(Hrs)	X (mg/L COD)	S (mg/L COD)
25.5	152.4	37.7
26.0	156.1	31.4
26.5	159.6	25.6
27.0	162.8	20.3
27.5	165.7	15.5
28.0	168.1	11.5
28.5	170.1	8.2
29.0	171.6	5.6
29.5	172.7	3.8
30.0	173.5	2.5
30.5	174.1	1.6
31.0	174.4	1.0
31.5	174.6	0.6
32.0	174.8	0.4
32.5	174.9	0.2
33.0	174.9	0.1
33.5	174.9	0.1
34.0	175.0	0.1
34.5	175.0	0.0
35.0	175.0	0.0
35.5	175.0	0.0
36.0	175.0	0.0
36.5	175.0	0.0
37.0	175.0	0.0
37.5	175.0	0.0
38.0	175.0	0.0
38.5	175.0	0.0
39.0	175.0	0.0
39.5	175.0	0.0
40.0	175.0	0.0
40.5	175.0	0.0
41.0	175.0	0.0
41.5	175.0	0.0
42.0	175.0	0.0
42.5	175.0	0.0
43.0	175.0	0.0
43.5	175.0	0.0
44.0	175.0	0.0
44.5	175.0	0.0
45.0	175.0	0.0
45.5	175.0	0.0
46.0	175.0	0.0
46.5	175.0	0.0
47.0	175.0	0.0
47.5	175.0	0.0
48.0	175.0	0.0
48.5	175.0	0.0
49.0	175.0	0.0
49.5	175.0	0.0
50.0	175.0	0.0

1d (10)

Time(Hrs)	X (mg/L COD)	S (mg/L COD)
0.0	50.0	250.0
0.5	51.9	246.8
1.0	53.9	243.4
1.5	56.0	240.0
2.0	58.2	236.4
2.5	60.4	232.7
3.0	62.7	228.8
3.5	65.1	224.8
4.0	67.6	220.7
4.5	70.2	216.4
5.0	72.8	212.0
5.5	75.6	207.4
6.0	78.4	202.6
6.5	81.4	197.7
7.0	84.5	192.5
7.5	87.6	187.3
8.0	90.9	181.8
8.5	94.3	176.2
9.0	97.8	170.3
9.5	101.4	164.3
10.0	105.1	158.1
10.5	109.0	151.7
11.0	112.9	145.1
11.5	117.0	138.3
12.0	121.2	131.4
12.5	125.5	124.2
13.0	129.9	116.8
13.5	134.4	109.3
14.0	139.0	101.7
14.5	143.7	93.8
15.0	148.5	85.9
15.5	153.3	77.9
16.0	158.1	69.8
16.5	163.0	61.7
17.0	167.8	53.7
17.5	172.5	45.9
18.0	177.0	38.3
18.5	181.4	31.0
19.0	185.4	24.3
19.5	189.0	18.4
20.0	192.0	13.3
20.5	194.5	9.2
21.0	196.4	6.0
21.5	197.7	3.8
22.0	198.6	2.3
22.5	199.2	1.4
23.0	199.5	0.8
23.5	199.7	0.5
24.0	199.8	0.3
24.5	199.9	0.2
25.0	199.9	0.1

Time(Hrs)	X (mg/L COD)	S (mg/L COD)
25.5	200.0	0.1
26.0	200.0	0.0
26.5	200.0	0.0
27.0	200.0	0.0
27.5	200.0	0.0
28.0	200.0	0.0
28.5	200.0	0.0
29.0	200.0	0.0
29.5	200.0	0.0
30.0	200.0	0.0
30.5	200.0	0.0
31.0	200.0	0.0
31.5	200.0	0.0
32.0	200.0	0.0
32.5	200.0	0.0
33.0	200.0	0.0
33.5	200.0	0.0
34.0	200.0	0.0
34.5	200.0	0.0
35.0	200.0	0.0
35.5	200.0	0.0
36.0	200.0	0.0
36.5	200.0	0.0
37.0	200.0	0.0
37.5	200.0	0.0
38.0	200.0	0.0
38.5	200.0	0.0
39.0	200.0	0.0
39.5	200.0	0.0
40.0	200.0	0.0
40.5	200.0	0.0
41.0	200.0	0.0
41.5	200.0	0.0
42.0	200.0	0.0
42.5	200.0	0.0
43.0	200.0	0.0
43.5	200.0	0.0
44.0	200.0	0.0
44.5	200.0	0.0
45.0	200.0	0.0
45.5	200.0	0.0
46.0	200.0	0.0
46.5	200.0	0.0
47.0	200.0	0.0
47.5	200.0	0.0
48.0	200.0	0.0
48.5	200.0	0.0
49.0	200.0	0.0
49.5	200.0	0.0
50.0	200.0	0.0

1d Alternate solution

Assume X can be represented by an average that is constant in PFR, X_{avg}

since $\Delta X = 135 \text{ mg CODx}/\ell \; (= Y(S_o - S))$

try $X_{avg} = \frac{135}{2} = 67.5 \text{ mg CODx}/\ell$

$$\int_{S_o}^{S} \left(\frac{K_s + S}{S}\right) dS = -\left(\frac{\hat{\mu} X}{Y}\right) \int_0^{\tau} d\tau$$

$$K_s \ln \frac{S}{S_o} + (S - S_o) = -\frac{\hat{\mu} X_{avg}}{Y} \tau$$

$$\tau = \frac{Y}{\hat{\mu} X_{avg}} \left(K_s \ln\left(\frac{S_o}{S}\right) + S_o - S\right)$$

If using this note that τ is very sensitive to X_{avg}

for $X_{avg} = 67.5$ & $S = 25 \text{ mg}/\ell$ COD

$$\tau_{PFR} = \frac{0.6}{2(67.5)} \left(50 \ln\left(\frac{250}{25}\right) + (250 - 25)\right) = 1.5 \text{ d}$$

$\simeq \tau_{CSTR}$ from 1b

for $X_{avg} = 100$

$$\tau_{PFR} = \frac{0.6}{2(100)} \left(50 \ln(10) + 225\right) = 1.0 \text{ d} \quad (33\% \text{ less})$$

Alternate solution to 1d, solve for τ, assuming X_{avg} = constant

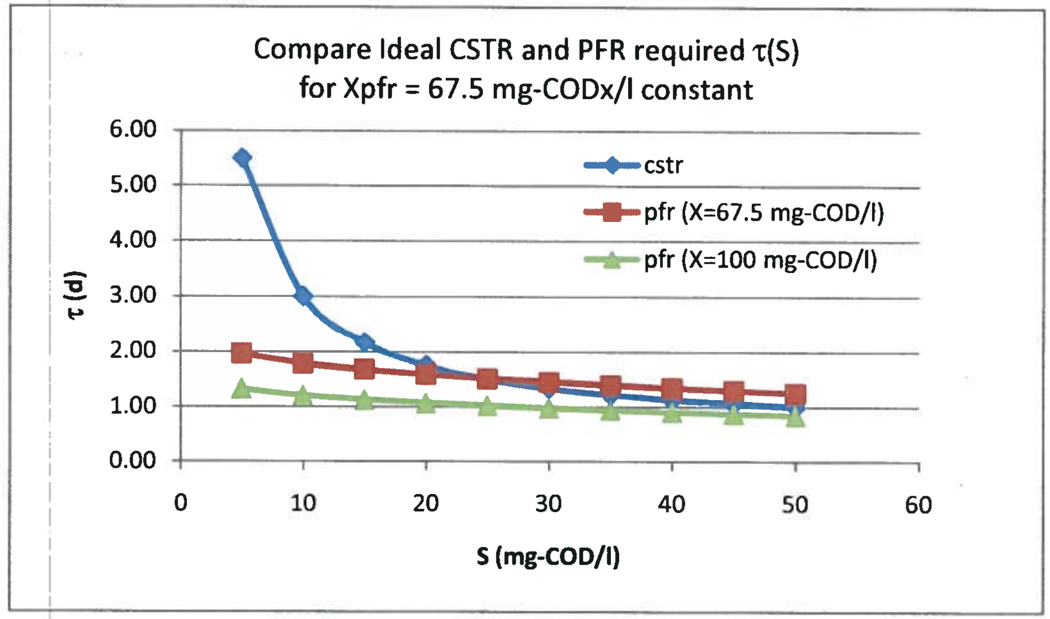
Ideal PFR assuming X_{avg} = 67.5 mg/l CODx = average between zero (inlet) and 135 (outlet)

X = 135/2 = 67.5 mg/l COD, So = 250 mg/l COD, Ks = 50 mg/l COD, Y = 0.6 gCODx/gCODs, $\mu_{max} = 2\ d^{-1}$

$\tau_{pfr} = (Y/(\mu_{max} * X_{avg}))*(Ks*\ln(So/S)+(So-S))$

$\tau_{cstr} = 1/\mu = (((Ks+S))/\mu_{max}*S)$

S mgCOD/l)	cstr	pfr (X=67.5 mg-COD/l)	pfr (X=100 mg-COD/l)
5	5.50	1.96	1.32
10	3.00	1.78	1.20
15	2.17	1.67	1.13
20	1.75	1.58	1.07
25	1.50	1.51	1.02
30	1.33	1.45	0.98
35	1.21	1.39	0.94
40	1.13	1.34	0.90
45	1.06	1.29	0.87
50	1.00	1.25	0.84
60	0.92	1.16	0.78
100	0.75	0.87	0.59



Outcome for PFR is strongly dependent on X_{avg}

For S < 25 mg/l COD, PFR has lower required hydraulic residence time (and volume), if X = constant at 67.5 mg/l

For Xpfr = 100 mgCOD/l, ideal PFR has lower residence time and volume for S < 100

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Problem 2

Influent COD = 300 mg/l

$r_S = -k*S$, $k = 5\ d^{-1}$

Values for τ (day)

COD Removal	CSTR	2-tank Cascade	4-tank cascade	PFR
0.85	1.13	0.63	0.49	0.38
0.99	19.80	3.60	1.73	0.92

formulas

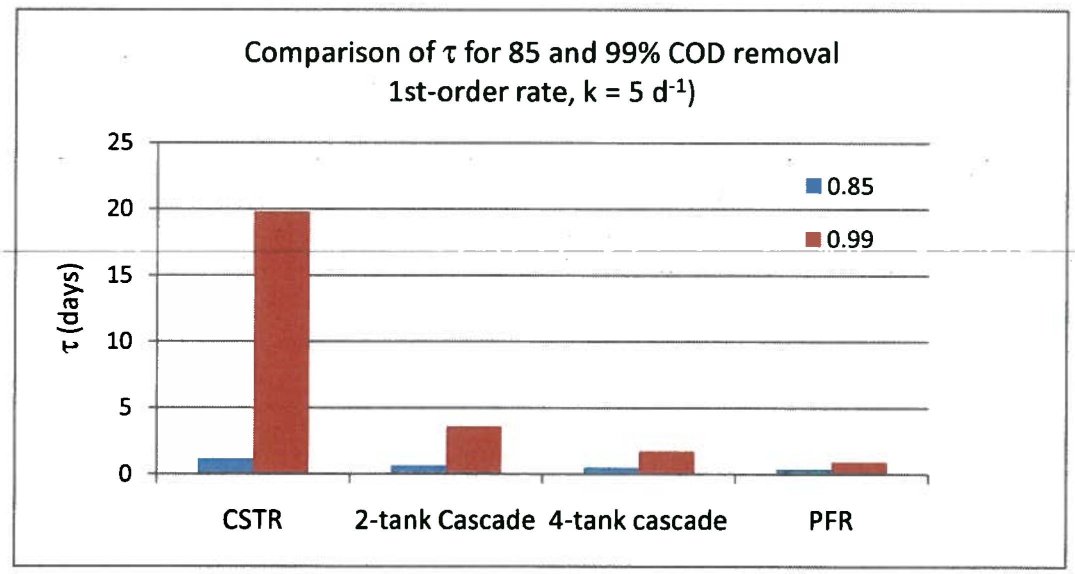
CSTR and Cascade

$n\tau = (n/k)((So/S)^{(1/n)}-1)$

n = 1, 2, and 4

PFR

$\tau = -(1/k)*\ln(S/So)$



3a. steady-state mass balance on X

$$-Q_W \cancel{X} + V(\mu - b)\cancel{X} = 0$$

$$\mu - b = \frac{Q_W}{V} = \frac{1}{\theta}$$

$$\mu = \frac{\hat{\mu} S}{K_s + S}$$

$$\frac{\hat{\mu} S}{K_s + S} = \frac{1}{\theta} + b = \frac{1 + b\theta}{\theta}$$

$$\hat{\mu} S = \frac{K_s + S}{\theta} + b(K_s + S)$$

$$S\left(\hat{\mu} - \left(\frac{1}{\theta} + b\right)\right) = \frac{K_s}{\theta} + bK_s = K_s\left(\frac{1}{\theta} + b\right)$$

$$S = \frac{K_s\left(\frac{1}{\theta} + b\right)}{\hat{\mu} - \left(\frac{1}{\theta} + b\right)} = \boxed{\frac{K_s(1 + b\theta)}{\hat{\mu}\theta - (1 + b\theta)}}$$

steadystate mass balance on S

$$QS_0 - \cancel{(Q - Q_W)}S - \cancel{Q_W}S + Vr_s = 0$$

$$-r_s = \frac{S_0 - S}{\tau} \quad \text{(same as no recycle)}$$

$$\frac{S_0 - S}{\tau} = \frac{\mu X}{Y}$$

$$X = \frac{Y(S_0 - S)}{\mu \tau} \qquad \mu = \frac{1}{\theta} + b$$

$$X = \frac{Y(S_0 - S)}{\frac{\tau}{\theta} + b\tau} = \frac{Y(S_0 - S)}{\frac{\tau}{\theta}(1 + b\theta)} = \boxed{\left(\frac{\theta}{\tau}\right)\frac{Y(S_0 - S)}{(1 + b\theta)}}$$

3b. find V, X for S = 25 mg/L COD

$$\frac{1}{\theta} = \frac{\hat{\mu} S}{K_s + S} - b = \frac{2(25)}{50+25} - 0.1 \quad d^{-1}$$

$$\theta = 1.8\,d = \frac{V}{Q_w}$$

$$V = 1.8\,d\,(300\ m^3/d) = \boxed{540\ m^3}$$

$$X = \frac{\theta}{\tau} \frac{Y(S_0 - S)}{(1 + b\theta)} = \frac{\theta Q}{V} \frac{Y(S_0 - S)}{(1+b\theta)}$$

$$X = \frac{1.8\,d\,(10^4\ m^3/d)}{540\ m^3} \frac{(0.6)(225)}{(1+0.1(1.8))}$$

$$X = 3800\ \frac{mg\ COD_x}{l}$$

c. Compare

CSTR	$\tau_{90\%}$ (d)	$V_{90\%}$ (m^3)	$X_{90\%}$ (mg COD_x/l)
recycle	0.054	1.5×10^4	135
no recycle	1.5	540	3800

$$\frac{V_{no\,recycle}}{V_{recycle}} = 28, \qquad \frac{X_{recycle}}{X_{no\,recycle}} = 28$$

cell recycle increases biomass by factor of 28 and reduces required reaction time and volume by factor of 28.