Generalized linear sampling method for active imaging of subsurface fractures

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Abstract

A theoretical foundation is developed for active seismic reconstruction of fractures endowed with spatially-varying interfacial condition (e.g. hydraulic fractures). The proposed indicator functional carries a superior localization property with no significant sensitivity to the fracture's contact condition, measurement errors, and illumination frequency. This is accomplished through the paradigm of the $F_\sharp$-factorization technique and the recently developed Generalized Linear Sampling Method (GLSM) applied to elastodynamics. The analysis of the well-posedness of the forward problem leads to an admissibility condition on the fracture's (linearized) contact parameters. This in turn contributes toward establishing the applicability of the $F_\sharp$-factorization method, and consequently aids the formulation of a convex GLSM cost functional whose minimizer can be computed without iterations. Such minimizer is then used to construct a robust fracture indicator function, whose performance is illustrated through a set of numerical experiments.

Keywords: inverse scattering, elastic-wave imaging, partially-closed fractures

1 Problem statement

With reference to Fig. 1(a), consider the elastic-wave sensing of a partially closed fracture $\Gamma \subset \mathbb{R}^3$ embedded in a homogeneous, isotropic, elastic solid endowed with mass density ρ and Lamé parameters μ and λ. The fracture is characterized by a heterogeneous contact condition synthesizing the spatially-varying nature of its rough and/or multi-phase interface. Next, let Ω denote the unit sphere centered at the origin. For a given triplet of vectors $\boldsymbol{d} \in \Omega$ and $\boldsymbol{q}_p, \boldsymbol{q}_s \in \mathbb{R}^3$ such that $\boldsymbol{q}_p \parallel \boldsymbol{d}$ and $\boldsymbol{q}_s \perp \boldsymbol{d}$, the obstacle is illuminated by a combination of compressional and shear plane waves

$$\boldsymbol{u}^{\rm f}(\boldsymbol{\xi}) \;=\; \boldsymbol{q}_p\, e^{{\rm i}k_p \boldsymbol{\xi}\cdot\boldsymbol{d}} \;+\; \boldsymbol{q}_s\, e^{{\rm i}k_s \boldsymbol{\xi}\cdot\boldsymbol{d}} \qquad (1)$$

propagating in direction $\boldsymbol{d}$, where k_p and $k_s = k_p\sqrt{(\lambda+2\mu)/\mu}$ denote the respective wave numbers. The interaction of $\boldsymbol{u}^{\rm f}$ with Γ gives rise to the scattered field $\boldsymbol{v} \in H^1_{\rm loc}(\mathbb{R}^3\backslash\Gamma)^3$, solving

$$\begin{aligned} &\nabla\cdot(\boldsymbol{C}:\nabla\boldsymbol{v}) + \rho\omega^2\boldsymbol{v} = \boldsymbol{0} \qquad && \text{in } \mathbb{R}^3\backslash\Gamma, \\ &\boldsymbol{n}\cdot\boldsymbol{C}:\nabla\boldsymbol{v} = \boldsymbol{K}(\boldsymbol{\xi})[\![\boldsymbol{v}]\!] - \boldsymbol{t}^{\rm f} && \text{on } \Gamma, \end{aligned} \qquad (2)$$

where $\omega^2 = k_s^2\mu/\rho$ is the frequency of excitation; $[\![\boldsymbol{v}]\!] = [\boldsymbol{v}^+ - \boldsymbol{v}^-]$ is the jump in $\boldsymbol{v}$ across Γ, hereon referred to as the fracture opening displacement (FOD); $\boldsymbol{C} = \lambda\,\boldsymbol{I}_2\otimes\boldsymbol{I}_2 + 2\mu\,\boldsymbol{I}_4$ is the fourth-order elasticity tensor; $\boldsymbol{I}_m\,(m=2,4)$ denotes the mth-order symmetric identity tensor; $\boldsymbol{t}^{\rm f} = \boldsymbol{n}\cdot\boldsymbol{C}:\nabla\boldsymbol{u}^{\rm f}$ is the free-field traction vector; $\boldsymbol{n} = \boldsymbol{n}^-$ is the unit normal on Γ, and $\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{\xi})$ is a *symmetric* and possibly *complex-valued* matrix of specific stiffness coefficients mapping the displacement jump to surface traction. The far-field patterns of the scattered waveforms i.e. $\boldsymbol{v}^\infty := \boldsymbol{v}_p^\infty \oplus \boldsymbol{v}_s^\infty$ – defined based on the asymptotic expansion of $\boldsymbol{v}$

$$\boldsymbol{v}(\boldsymbol{\xi}) = \frac{e^{{\rm i}k_p r}}{4\pi(\lambda+2\mu)r}\,\boldsymbol{v}_p^\infty(\hat{\boldsymbol{\xi}}) + \frac{e^{{\rm i}k_s r}}{4\pi\mu r}\,\boldsymbol{v}_s^\infty(\hat{\boldsymbol{\xi}}) + O(r^{-2}),$$

as $r := |\boldsymbol{\xi}| \to \infty$, are then recorded over the unit sphere of observation directions $\hat{\boldsymbol{\xi}}$.

Theorem (well-posedness). Assume that $\boldsymbol{t}^{\rm f} \in H^{-1/2}(\Gamma)^3$ and that $\boldsymbol{K} \in L^\infty(\Gamma)^{3\times3}$ is symmetric such that $\Im\boldsymbol{K} \leqslant \boldsymbol{0}$ on Γ, i.e. that $\overline{\boldsymbol{\theta}}\cdot\Im\boldsymbol{K}(\boldsymbol{\xi})\cdot\boldsymbol{\theta} \leqslant 0$, $\forall\boldsymbol{\theta} \in \mathbb{C}^3$ and a.e. on Γ. Then problem (2) has a unique solution that continuously depends on $\boldsymbol{t}^{\rm f} \in H^{-1/2}(\Gamma)^3$.
proof. see [1, Theorem 3.2].

2 Elements of the inverse solution

For given compressional and shear wave densities $\boldsymbol{g}_p(\boldsymbol{d}) \parallel \boldsymbol{d}$ and $\boldsymbol{g}_s(\boldsymbol{d}) \perp \boldsymbol{d}$, $\boldsymbol{d} \in \Omega$, the elastic Herglotz wave function is defined as

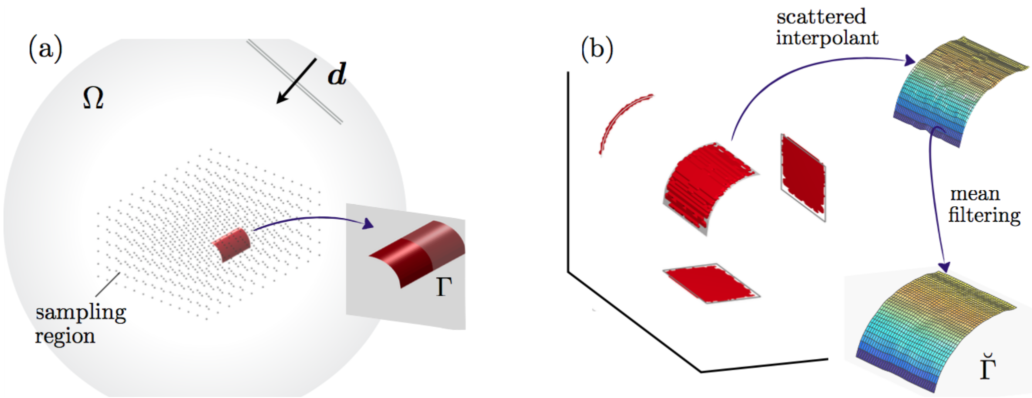

Figure 1: Elastic-wave imaging of a heterogeneous fracture Γ: (a) sensing configuration, and (b) 3D GLSM indicator thresholded at 10% and thus-recovered fracture surface $\breve{\Gamma}$.

$$\mathbf{u}_{\boldsymbol{g}}(\boldsymbol{\xi}) := \int_{\Omega} \Big\{ \boldsymbol{g}_p(\boldsymbol{d})\, e^{\mathrm{i}k_p \boldsymbol{d}\cdot\boldsymbol{\xi}} \oplus \boldsymbol{g}_s(\boldsymbol{d})\, e^{\mathrm{i}k_s \boldsymbol{d}\cdot\boldsymbol{\xi}} \Big\}\, \mathrm{d}S_{\boldsymbol{d}},$$

where $\boldsymbol{\xi} \in \mathbb{R}^3$. In this setting, the far-field operator $F: L^2(\Omega)^3 \to L^2(\Omega)^3$ is defined by

$$F(\boldsymbol{g}) \;=\; \boldsymbol{v}^{\infty}_{\boldsymbol{g}_{\Omega}}, \tag{3}$$

where $\boldsymbol{v}^{\infty}_{\boldsymbol{g}_{\Omega}}$ is the far-field pattern of $\boldsymbol{v} \in H^1_{\text{loc}}(\mathbb{R}^3 \backslash \Gamma)^3$ solving (2) with data $\boldsymbol{u}^{\mathrm{f}} = \mathbf{u}_{\boldsymbol{g}}$. Based on this, the self-adjoint operator $F_\sharp\colon L^2(\Omega)^3 \to L^2(\Omega)^3$ is defined by

$$F_\sharp := |\Re F| + \Im F, \tag{4}$$

$$\Re F = \tfrac{1}{2}(F + F^*), \qquad \Im F = \tfrac{1}{2i}(F - F^*).$$

It is then shown that both F and $F_\sharp$ possess the following decompositions

$$F = \mathscr{H}^* T \mathscr{H}, \quad F_\sharp = \mathscr{H}^* T_\sharp \mathscr{H}. \tag{5}$$

Here, the Herglotz operator $\mathscr{H}\colon L^2(\Omega)^3 \to H^{-1/2}(\Gamma)^3$ is given by

$$\mathscr{H}(\boldsymbol{g}) \;:=\; \boldsymbol{n}\cdot\boldsymbol{C}:\nabla\mathbf{u}_{\boldsymbol{g}} \quad \text{on} \quad \Gamma, \tag{6}$$

whose adjoint operator $\mathscr{H}^*: \tilde{H}^{1/2}(\Gamma)^3 \to L^2(\Omega)^3$ is shown to be compact and injective; The middle operator $T\colon H^{-1/2}(\Gamma)^3 \to \tilde{H}^{1/2}(\Gamma)^3$ (resp. $T_\sharp$) is governed by the contact law at the fracture interface, and given by

$$T(\boldsymbol{t}^{\mathrm{f}})(\boldsymbol{\xi}) \;:=\; [\![\boldsymbol{v}(\boldsymbol{\xi})]\!], \qquad \boldsymbol{\xi} \in \Gamma, \tag{7}$$

(resp. [1, Remark 4]). Now, the following properties form the bedrock of the GLSM's theorem and its affiliated indicator, namely: (i) the ranges of $\mathscr{H}^*$ and $F_\sharp^{1/2}$ coincide, and (ii) both operators T and $T_\sharp$ are continuous and coercive i.e.

$$\big(\boldsymbol{\varphi}, T_\sharp(\boldsymbol{\varphi})\big)_{H^{-\frac{1}{2}}(\Gamma)} \;\geqslant\; c\,\|\boldsymbol{\varphi}\|^2_{H^{-\frac{1}{2}}(\Gamma)},$$
$$|\langle \boldsymbol{\varphi}, T(\boldsymbol{\varphi})\rangle| \;\geqslant\; \mathrm{c}\,\|\boldsymbol{\varphi}\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \;\; \forall \boldsymbol{\varphi} \in H^{-1/2}(\Gamma)^3$$

where $\mathrm{c}, c > 0$ are independent of $\boldsymbol{\varphi}$.

3 GLSM criteria for imaging

The essential idea behind GLSM stems from the particular nature of an approximate solution $\boldsymbol{g}$ to the far-field equation

$$F^\delta \boldsymbol{g} \;=\; \boldsymbol{\Phi}^{\infty}_{L}, \qquad \|F^\delta - F\| \;\leqslant\; \delta, \tag{8}$$

where $\delta > 0$ is a measure of noise in data, and $\boldsymbol{\Phi}^{\infty}_{L}$ is the far-field pattern of a trial radiating field, see [1, Definition 2]. In this setting, the behavior of $\boldsymbol{g}$ in the sampling region is exposed by characterizing the range of $\mathscr{H}^*$, which then forms the basis for approximating the characteristic function of a hidden fracture. In this vein, let us define the GLSM cost functional by

$$\mathfrak{J}^\delta_\alpha(\boldsymbol{\Phi}^{\infty}_{L}; \boldsymbol{g}) \;:=\; \|F^\delta \boldsymbol{g} - \boldsymbol{\Phi}^{\infty}_{L}\|^2 \;+\; \alpha\big(|(\boldsymbol{g}, B^\delta \boldsymbol{g})| + \delta\,\|\boldsymbol{g}\|^2\big), \;\; \boldsymbol{g} \in L^2(\Omega)^3, \tag{9}$$

where $\alpha > 0$ and B^δ denotes either F^δ or $F^\delta_\sharp$. Assuming that B^δ is compact, $\mathfrak{J}^\delta_\alpha$ has a minimizer $\boldsymbol{g}^L_{\alpha,\delta} \in L^2(\Omega)^3$ satisfying

$$\lim_{\alpha\to 0}\limsup_{\delta\to 0}\, \mathfrak{J}^\delta_\alpha(\boldsymbol{\Phi}^{\infty}_{L}; \boldsymbol{g}^L_{\alpha,\delta}) \;=\; 0. \tag{10}$$

In the case where $B^\delta = F^\delta_\sharp$, the cost functional (9) is convex and that its minimizer is obtained non-iteratively.

Theorem (main). $\boldsymbol{\Phi}^{\infty}_{L} \in \mathit{Range}(\mathscr{H}^*) \iff \big\{\limsup_{\alpha\to 0}\limsup_{\delta\to 0}(|(\boldsymbol{g}^L_{\alpha,\delta}, B^\delta \boldsymbol{g}^L_{\alpha,\delta})| + \delta\|\boldsymbol{g}^L_{\alpha,\delta}\|^2) < \infty, \iff \liminf_{\alpha\to 0}\liminf_{\delta\to 0}(|(\boldsymbol{g}^L_{\alpha,\delta}, B^\delta \boldsymbol{g}^L_{\alpha,\delta})| + \delta\|\boldsymbol{g}^L_{\alpha,\delta}\|^2) < \infty\big\}$.

Based on this, a robust GLSM criterion

$$I^{\mathscr{G}}(L) \;:=\; \big[|(\boldsymbol{g}^L_{\alpha,\delta}, B^\delta \boldsymbol{g}^L_{\alpha,\delta})| + \delta\,\|\boldsymbol{g}^L_{\alpha,\delta}\|^2\big]^{-1/2}$$

is designed for the reconstruction of heterogeneous fractures, as illustrated in Fig. 1 (b).

References

[1] F. Pourahmadian and B. B. Guzina and Houssem Haddar, Generalized linear sampling method for elastic-wave sensing of heterogeneous fractures, *Inverse Problems*, 33(5) 055007.