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IDENTIFYING NORMAL MODES OF A NONLINEAR SYSTEM

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ABSTRACT

Nonlinear normal modes (NNMs) provide a useful tool for extending modal analysis to nonlinear systems and provide a foundation to obtain reduced order models. This paper employs two different approaches in identifying NNMs of a nonlinear structure using experimentally measured data. In the first approach, equivalent linear models associated with the nonlinear system are identified at different response amplitudes and using these models the NNMs are determined. The second method uses the fact that the deformed shape of a nonlinear system near its resonance frequency accurately represents it NNM. The later approach is capable of identifying the normal modes of complex structures with global nonlinearities and is particularly useful when defining an equivalent linear system is not an easy task or practically impossible. In an experimental case study a beam structure with frictional support boundary exhibiting micro-slip at its boundary is considered. The beam is excited using a harmonic force and its response is recorded for further process. NNMs are then obtained using the two described approaches and it is shown the results of two methods are similar. However it is important to note that in the second approach no prior knowledge regarding the system nonlinearity is required and it can be used in identification of the nonlinear effects with high accuracy.

1. INTRODUCTION

A common approach in reducing the order of a nonlinear continuous or discrete model is to employ the mode shapes of a linear base structure. Provided the system under consideration is weakly nonlinear, and the damping effects (in linear or nonlinear form) are sufficiently small, the natural frequencies are well separated and the structure is in a resonant condition, the system can be regarded as a single degree of freedom system; the approach is known as single mode method. The reason for employing the mode shapes of the base linear system for reduction may be their simple calculation. Moreover, these modes are rather good approximations of the actual vibrational modes of structure in the resonant condition but they may not be adequately precise.

Comparison between the results of single mode method and that of numerical integration shows the single mode method may lead to considerable errors [1]. Consequently, employing techniques to improve the single mode method deficiencies and developing new concepts to solve these problems efficiently are inevitable.

Rosenberg pioneered the development of NNMs concept and in his studies he introduced different methods of extracting them [2]. Szemptinska-Stupincka [3] revealed that the mode of vibration in resonance condition can be a good approximation of NNMs. She employed the Ritz method to calculate these modes analytically as a function of modal amplitude. The nonlinear eigen-value problem encountered in her proposed method to determine the modal parameters was solved by a numerical procedure. Jezequel [4] and Setio et al. [5, 6] developed a procedure to determine the modal parameters by curve fitting the forced responses in the neighborhood of a resonance point. Their method combines the theory introduced by Szemptinska-Stupincka and an equivalent linearization approach of Iwan [7]. Nayfeh et al. [8] applied the invariant manifold and multiple scales methods to study the nonlinear modes of one-dimensional continuous systems. Nobari et al. [9] used the sensitivity analysis for prediction of the NNMs. A comprehensive literature survey about NNMs has been presented by Vakakis [10]. The review of the recent contributions in this field can be found in [11, 12].

This paper employs the concept introduced by Szemptinska-Stupincka [3] and the equivalent linearization method [7] to identify the single nonlinear normal mode using experimental test data. The method assumes the nonlinear normal mode is a combination of the base linear system normal modes. By using the measured responses, the contribution of each linear mode shape in nonlinear normal mode is determined.

In an experimental case study this paper considers the nonlinear behavior of a beam with frictional contact support. In the study a constant normal force is applied on the contact interface while the beam response level at the driving point is kept constant. This is achieved by controlling the excitation force in a small frequency bandwidth near the first natural frequency of the system. This procedure is then repeated for several vibration amplitudes. The increase in vibration amplitudes creates different behaviors in the contact interface ranging from sticking to micro-slips and finally macro-slips at higher response levels.

The structure nonlinearity is local and it is a common approach to introduce an equivalent linear system for the structure. The measured FRFs reveal the natural frequencies of the equivalent linear system in different vibration amplitudes. The mode shapes of the equivalent linear system at each vibration amplitude is commonly regarded as the nonlinear normal modes. In this study the NNMs are identified using both the equivalent linear system approach and the proposed method of this paper and the results are compared with each other.

The remaining of this paper is organized as followings. In the next section the proposed method of identifying NNMs is described. Next, an experimental case study is considered and linear and nonlinear FRFs of the test setup at different vibration levels are measured. By using the experimental results, the NNMs are identified using the two approaches mentioned above. Then, conclusions are drawn and references are presented.

2. NONLINEAR NORMAL MODE IDENTIFICATION

The equation of motion of a continuous nonlinear vibrating system can be considered in the following form,

$$\frac{\partial^2 w}{\partial t^2}+L\left[w\left(x,t\right)\right]+N\left[w\left(x,t\right)\right]=f(x,t). \tag{1}$$

The operators L and N describe the linear and nonlinear parts of the equation of motion, respectively. The system can be subjected to general non-homogeneous boundary conditions; therefore, problems with nonlinear boundary effects can be inspected. In the present study, the external excitation force $f\,(x,t)$ is considered to be a single harmonic force and the excitation frequency is chosen close to the natural frequencies of the base linear system which is obtained by neglecting the nonlinear part of equation (1).

The mode of vibration in resonant condition is a good estimation of the nonlinear normal mode [3]. The idea in this paper is to construct the NNM by using appropriate number of mode shapes of the base linear system and employing measured time responses. The response of the system described by equation (1) can be expressed using its first n modes, $\varphi_i(x)$ $i=1,2,..,$ of the base linear system as,

$$w(x,t)=\sum_{i=1}^{n}\varphi_i(x)q_i(t). \tag{2}$$

In equation (2), $q_i\,(t)$ is the generalized coordinate. One may employ the above relation and define a direct relation between measured accelerations and the corresponding generalized coordinates. Logically, the number of mode shapes used in equation (2) may not exceed the number of independently measured vibration signals. It is possible to initially estimate the number of mode shapes contributing in the dynamic response of the structure and employ sufficient number of accelerometers in the measurement setup. The generalized coordinate vector can be calculated using the measured accelerations at j points, $j\geq n$, and the linear mode shape matrix as,

$$\ddot{\mathbf{q}}(\mathbf{t})=\begin{Bmatrix}\ddot{q}_1(t)\\ \ddot{q}_2(t)\\ \vdots\\ \ddot{q}_n(t)\end{Bmatrix}=\begin{bmatrix}\varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1)\\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_1(x_j) & \varphi_2(x_j) & \cdots & \varphi_n(x_j)\end{bmatrix}^{+}\begin{Bmatrix}\ddot{w}(x_1,t)\\ \ddot{w}(x_2,t)\\ \vdots\\ \ddot{w}(x_j,t)\end{Bmatrix}=\mathbf{\Phi}^{+}\ddot{\mathbf{w}} \tag{3}$$

The superscript $()^{+}$ refers to pseudo inverse of the matrix. The excitation force is considered to be simple harmonic; therefore the response contains the same harmonic as the excitation force and its multiples due to nonlinear effects in the structure. Considering that the excitation frequency is ω, a Fourier series of the following form can be fitted to every generalized acceleration signal,

$$\ddot{q}_i(t) = \sum_{m=1}^{M} A_{mi} sin(m\omega t + \psi_{mi}) = A_{1i} sin(\omega t + \psi_{1i}) + ..., \ i = 1, 2, 3. \tag{4}$$

where coefficients A_{mi} and ψ_{mi} are obtained from data fitting (see [14] for more details). The nonlinear normal mode is regarded as the first harmonic response of the system. This is a valid assumption if the structure has well separated natural frequencies and the nonlinearity and damping characteristics are weak. In these circumstances the nonlinear mode shapes are almost real and the generalized coordinates are mono-phase. Therefore the subscript *'i'* is omitted from ψ_{1i} in equation (4). The validity of this assumption is shown later in this paper using experimental results. The generalized coordinated $q_i(t)$ can be obtained by integrating equation (4) twice which leads to:

$$q_i(t) = A_{0i} - \sum_{m=1}^{M} \frac{A_{mi}}{(m\omega)^2} \sin(m\omega t + \psi_m) = A_{0i} - \frac{A_{1i}}{\omega^2} \sin(\omega t + \psi_1) + ..., \ i = 1, 2, 3. \tag{5}$$

The coefficient A_{0i} is zero when the nonlinearity is odd [14]. Substituting equation (5) in equation (2) one may obtain the following,

$$w(x,t) \simeq -\left(\sum_{i=1}^{n} \frac{A_{1i}\varphi_i(x)}{\omega^2} \right) \sin(\omega t + \psi_1). \tag{6}$$

And finally we define the nonlinear normal mode as follows,

$$\tilde{\varphi}_i(x,a) \simeq \frac{1}{|A_1|} \sum_{i=1}^{n} A_{1i}\varphi_i(x). \tag{7}$$

where $|A_1|$ is the norm of vector formed by coefficients A_{1i} and a is a measure of vibration level amplitude. Using mass normalized mode shapes of the base linear system, one may determine the normalized NNMs from equation (7). The concept which is used in definition of the NNMs in this paper is the same as that of Szemptinska-Stupincka [3] but the procedure used to obtain them is different. The nonlinear normal modes defined in equation (7) are functions of response amplitude *'a'* because different values are obtained for coefficients A_{1i} at different response amplitude levels (see equation (4)).

In the next section an experimental case study is considered and the first nonlinear normal mode is obtained using the above described procedure. In order to validate the accuracy of the method presented in this paper, the NNM are also obtained by using the concept of equivalent linearized system.

3. EXPERIMENTAL CASE STUDY

A steel beam of length *L=600 mm*, width *b=40 mm* and thickness *h=5 mm* clamped at one end and supported using a frictionally contact interface at the other end is employed in this experimental study. The frictionally contact boundary condition is provided by a pin welded to the beam end and is allowed to slip on a steel block as shown in figure 1. The pin has a radius of *r=6mm* and its length is equal to the beam width. A constant normal force is applied to the pin using suspended mass blocks.

The contact interface exhibits different behaviors depending on the vibration amplitude level *a.* In low excitation amplitude levels the interface is in stick regime and behaves linearly. Nonlinear behavior arises due to micro and macro slippage as the excitation force level increases. In the followings the behavior of contact is investigated at different vibration amplitudes.

The structure is excited using a *B&K4200* mini shaker attached through a stinger to the structure at distance *S=550 mm* from the clamped end. A *B&K8200* force transducer is placed between stinger and the structure. The structural responses are measured using three accelerometers mounted on the beam at locations x_1*=550 mm*, x_2*=300 mm* and *x3=100 mm* (measured from the beam clamped end). Figure 1 shows the transducer arrangement.

Dynamic responses of the structure are measured while the contact interface is normally loaded with 15 *kg* mass blocks, i.e. the blocks are equivalent to *147.15N* gravity force. Initially the structure is excited using a low level random force, ensuring the frictional contact interface is in stick regime, and the linear frequency response functions are measured. Figure 2 shows the driving point frequency response function recorded at low level of random excitation force. The corresponding natural frequencies are tabulated in table 1. The natural frequencies of Table 1 are used in this study to form the base linear system of the test structure.

ω_1	ω_2	ω_3
52.85	164.00	330.25

Table 1: Resonance frequencies at low level random excitations (Hz)

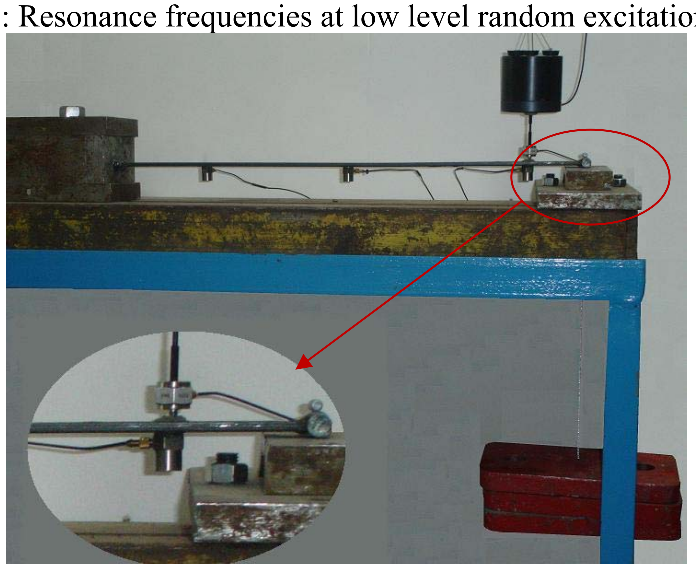

Figure 1. The test set-up of frictionally supported beam

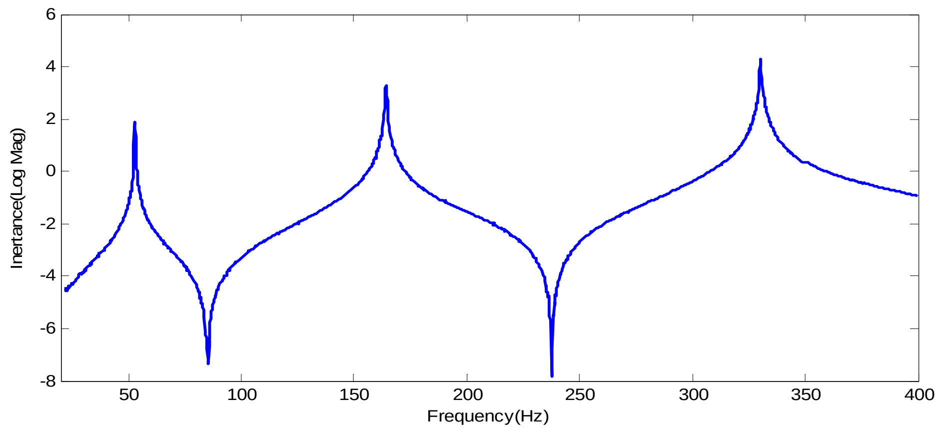

Figure 2. Driving point frequency responses at low level excitations

Next we turn our attention to investigate the structural response when nonlinear mechanisms develop in the contact interface. There are two common approaches in experimentally determining the frequency response curves of a nonlinear structure, namely measuring the FRFs at constant force level or at constant response level [13]. In this paper, the later approach is used. The force and response signals are recorded at different excitation frequencies while the response amplitude level is kept constant. Having the excitation force and response signals measured the FRFs can be constructed. Keeping the response amplitude levels constant in a small frequency band near the first natural frequency insures that the nature of the nonlinear mechanisms do not change in different frequencies even if the nonlinearity is velocity or acceleration dependent. At each response amplitude level an equivalent linearized system is fitted on the measured FRFs as is shown in Figure 3. From these FRFs the natural frequency of the equivalent linearized system can be identified. The bandwidth of selected points in curve fitting around each resonance point is 1Hz. Figure 3 shows the fitted frequency response curves at the driving point in different response amplitudes.

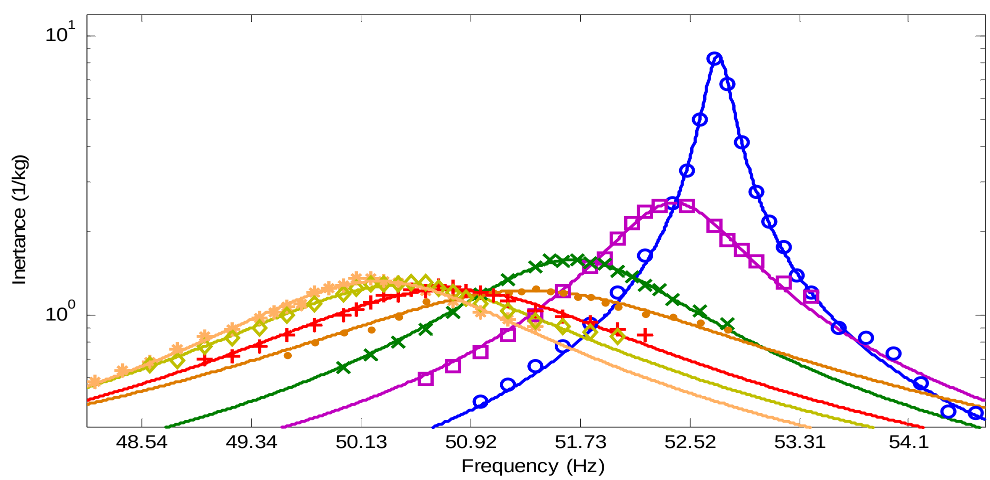

Figure 3. Measured and fitted frequency responses at response amplitudes of 10 m/s^2(circles), 20 m/s^2(squares), 30 m/s^2(×),40 m/s^2(points), 50 m/s^2(+),60 m/s^2(diamonds), and 70 m/s^2(*).

The frequency responses shown in Figures 3 resemble the equivalent linearized behavior of the structure. This is due to the fact that the response amplitude levels are kept constant during measurement of each frequency responses. The frequency responses shown in figure 3 are used in next section and an equivalent linearized system is constructed at each response amplitude level.

4. IDENTIFICATION OF NONLINEAR NORMAL MODES

In this section the first nonlinear normal mode of the structure shown in figure 1 is obtained by using the approach described in this paper and the equivalent linearized system approach. First a mathematical representation of the structure needs to be defined. For this purpose, the dynamic behavior of a fixed-frictionally supported beam as shown in figure 4 is considered. The structure is modeled using Euler-Bernoulli beam theory. The beam has a modulus of elasticity of E, cross sectional moment of inertia of I, mass density of ρ, cross sectional area of A, and length of L. The structure is excited using a concentrated force $f(t)$ at a distance S measured from its fixed end. A normal constant force F is applied on the frictional support. The applied normal force is large enough to restrict the lateral movement of the beam at frictional support but rotation is allowed at this end.

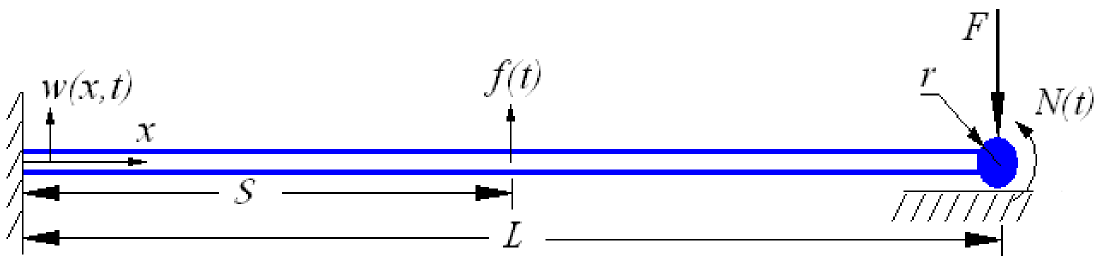

Figure 4. A slender beam with frictional contact boundary condition

The lateral vibration of a uniform continuous beam is governed using a nonlinear partial differential equation as:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=f(t)\delta(x-S)-N(t)\delta^{'}(x-L) \tag{8}$$

There is an offset r between neutral axis of the beam and the frictional force line of action which causes a bending moment at the boundary. So the nonlinear effects are included in the system via frictional contact moment $N(t)$ at the boundary. This bending moment is applied to the beam model using $\delta^{'}(x)$ which is the spatial derivative of the Dirac delta function $\delta(x)$ commonly known as a spatial unit doublet [16]. The frictional moment $N(t)$ takes into account the linear and non-linear characteristics of the contact interface. The nonlinear characteristics involve both nonlinearities in stiffness and damping.

First, identification of the NNM by using the concept of equivalent linearized system is considered. The nonlinearity in the structure shown in figure 1 is local. Therefore it is possible to define a physical equivalent linearized system [9, 15] for the structure as:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{9}$$

$$w(0,t)=0,\ \frac{\partial w(0,t)}{\partial x}=0,\ w(L,t)=0,\ EI\frac{\partial^2 w(L,t)}{\partial x^2}=k_\theta(a)\frac{\partial w(L,t)}{\partial x} \tag{10}$$

The flexural spring $k_\theta(a)$ is the equivalent stiffness of the frictional support at each vibration amplitude level. The mode shapes obtained from classical Euler-Bernoulli beam theory and boundary conditions defined in equations (9-10), will be used to construct the nonlinear normal modes of the system under consideration at different vibration amplitudes a.

In the measurements, the frequency responses near the first resonance point were recorded at different amplitudes. Using these measured responses one may identify the first natural frequency of the corresponding linear system. The natural frequencies are identified by curve fitting each frequency responses. The amplitude dependent natural frequencies obtained from the measurements are tabulated in table 2. In this table the corresponding flexural spring coefficients $k_\theta(a)$ are also shown. The stiffness $k_\theta(a)$ at each amplitude level a is identified by solving the characteristic equation of linear problem defined in equations (9-10). In development of the characteristic equation, the beam is divided into four parts; first part spans between fixed end and the accelerometer three at x_3, parts two and three are located between the three accelerometers, and part four is located between driving point and frictional support. The characteristic equation of the system is developed by considering the boundary conditions and the compatibility requirements at the interface of each two parts. In deriving the compatibility equations it is assumed that the displacements and slopes at the interface of each two parts are continuous but the shear forces and bending moments may abruptly change due to mass and inertia effects of the accelerometers and the force transducer.

a, m/s^2	**$\omega_1(a)$,Hz**	$k_\theta(a)$, **kN.m/rad**
Linear	52.85	197.4
10	52.79	192.8
20	52.30	167.1
30	51.62	125.4
40	50.63	73.6
50	50.55	71.4
60	50.30	65.7
70	50.11	58.7

Table 2: Changes of 1st natural frequency and the support stiffness with respect to the vibration amplitude

The normal modes of the above identified equivalent linearized systems are different at different response amplitude levels. They can be considered as nonlinear normal modes [9]. The amplitude dependent nonlinear normal modes $\omega_i(a), \tilde{\phi}_i(x,a)$ are compared with the first mode shape of the base linear system in figure 5. As it was stated in previous section, the linear natural frequencies tabulated in table 1 are used and the base linear system is constructed. We scaled the obtained mode shapes in order to be able to show all of nonlinear modes in one figure. The results presented in figure 5 indicate that by increasing response amplitude level the nonlinearity at the contact interface and hence the deviation of the nonlinear normal modes from linear normal modes increases.

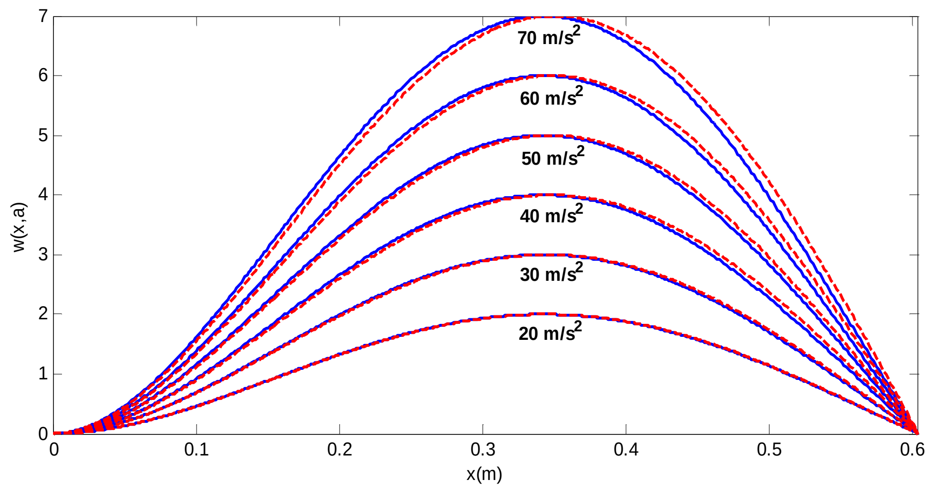

Figure 5. Comparing NNMs obtained by equivalent linearized system (red) and the first mode of the base linear system (blue)

Next identification of the NNMs by using the approach presented in this paper is considered. It is initially assumed only the first three normal modes of the base linear system contribute in the dynamic response as the structure has well separated modes and we have employed only three accelerometers in the measurement setup. Following this assumption is validated by showing that response is dominated only by the first two modes. Figure 6 shows the generalized coordinates $q_i(t)$ obtained from equation (3) at a=70m/s^2. At this amplitude we expect higher modes to have the highest contributions in the response compared to other vibration amplitudes. As it can be seen in figure 6, the response is dominated by the first mode contribution, the effect of second mode in response is marginal, and the third mode contribution in the response is approximately zero. Also the phase differences between different generalized coordinates are negligible, i.e. less than 4° so our assumption which considered the mode shapes real is true.

The response at the driving point is reconstructed using only the first two modes and is compared with the measured accelerations in figure 7. There is an excellent agreement between the measured and predicted responses ensuring the accuracy of the calculated generalized accelerations.

The results shown in figure 7 indicate that two modes are sufficient to expand the response. Therefore, by setting *n* in equation (7) equal to 2, the first NNM in each amplitude level can be identified by using the proposed method in this paper. The mode shapes used in equation (7) belong to the base linear system and are obtained by updating the system described in equations (9-10) using the linear natural frequencies presented in table 1. Also the time domain responses corresponding to resonant points of each FRFs are used in equation (4). Figure 8 compares the NNMs of figure 5 and the ones obtained by the method proposed in this paper. There is an excellent agreement between these two sets of results.

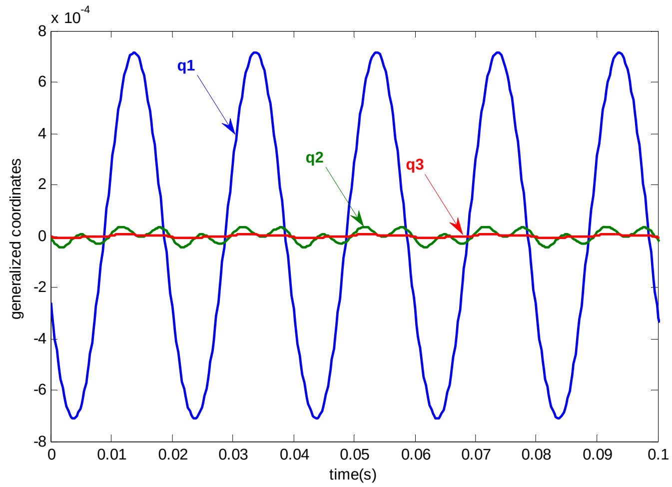

Figure 6. The generalized coordinates at a=70m/s^2

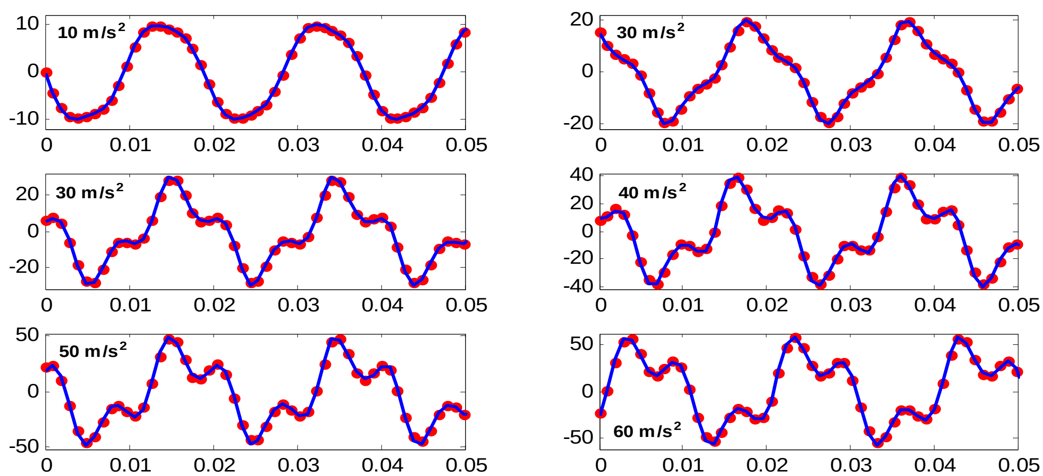

Figure 7. Accelerations at the driving point, measured (solid line) and reconstructed (circles)

5. CONCLUSION

A method was presented for identification of the nonlinear normal modes. The method is based on the assumption that a nonlinear normal mode can be expanded by using the normal modes of the base linear system. By using the measured responses, the contribution of each linear mode shape in nonlinear normal mode was calculated. This approach is capable of identifying the normal modes of complex structures with global nonlinearities and is particularly useful when defining an equivalent linear system is not an easy task or practically impossible. Moreover, it is important to note that in this method no prior knowledge regarding the system nonlinearity is required and it can be used in identification of the nonlinear effects with high accuracy. An experimental case study of a clamped-frictionally contact beam was considered. The experimental FRFs were measured at different response amplitude levels. Having the experimental FRFs, the nonlinear normal modes were identified at different response amplitudes by using the concept of equivalent linearized system. Also, the measured responses in time domain at excitation frequencies corresponding to resonant points of FRFs were used and the NNMs were identified using the method presented in this paper. The two sets of results showed an excellent agreement.

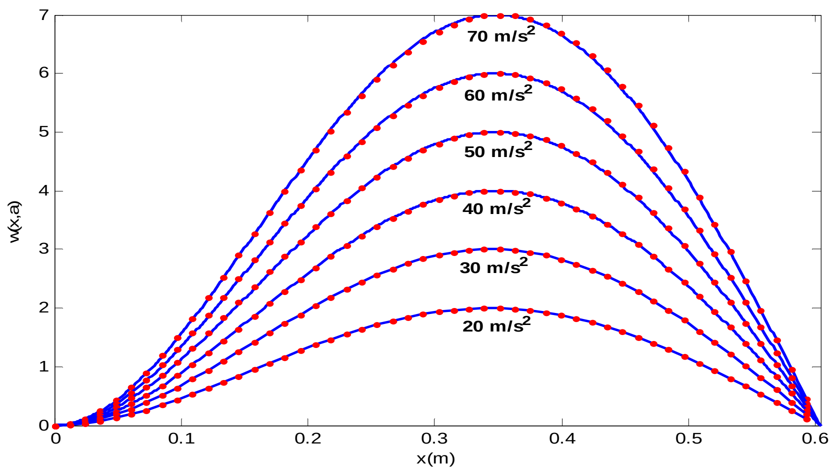

Figure 8. First NNMs: the equivalent linearized systems (blue), the method proposed in this paper (red)

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