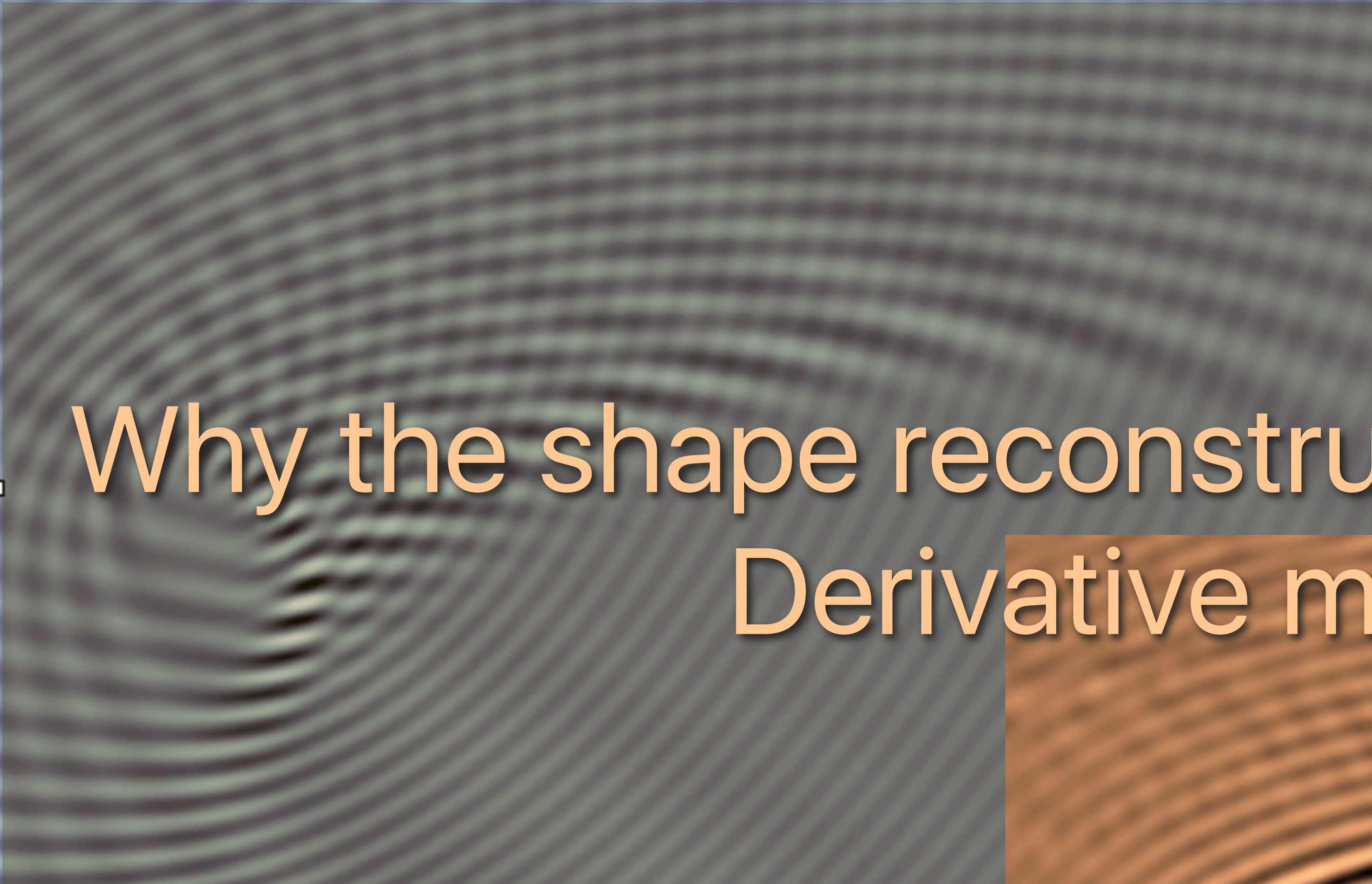
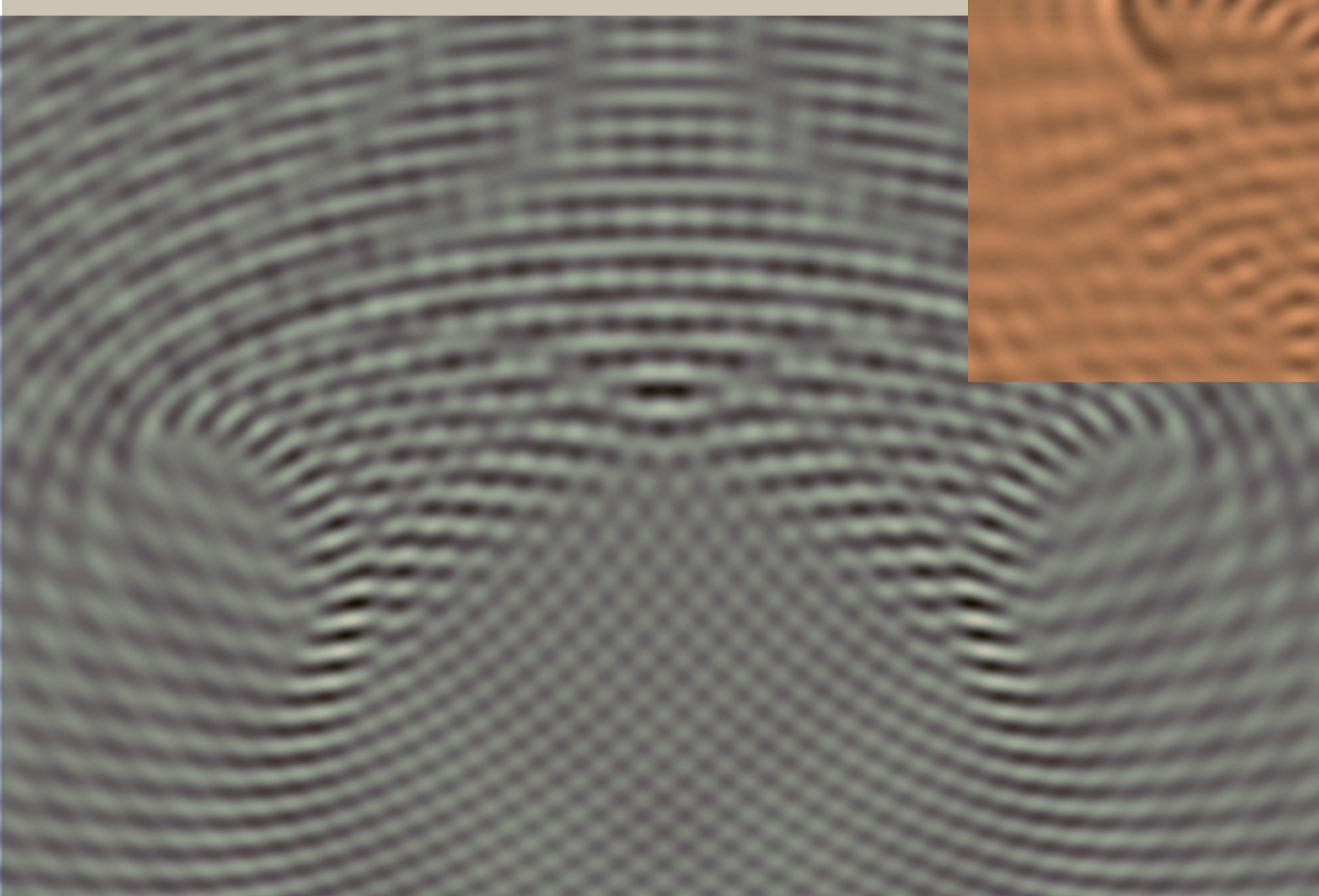
Why the shape reconstruction by Topological Derivative may work

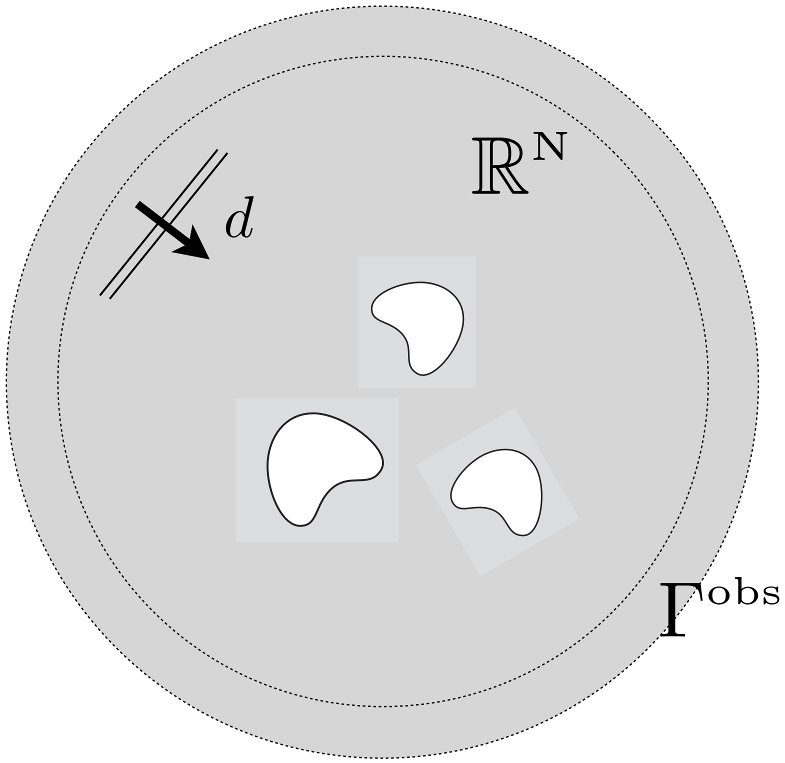




Fatemeh Pourahmadian

Bojan Guzina

TS



1994-
Eschenauer, Schumacher, Sokolowski, Zochowski, Garreau, Novotny, Allaire, ...

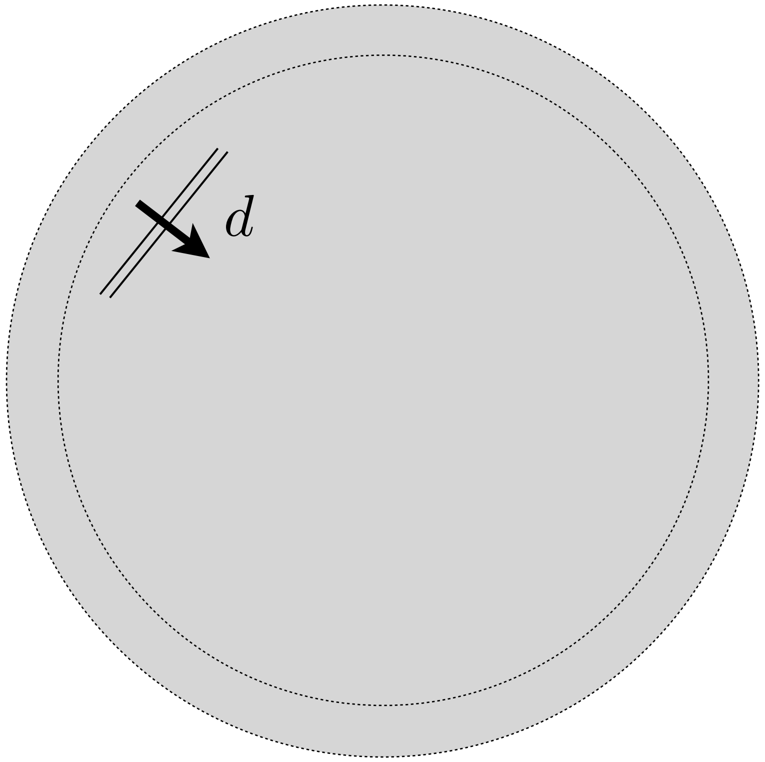
Shape optimization

1998-
Vogelius, Volkov, Ammari, Kang, Moskow, Masmoudi, Amstutz, ...

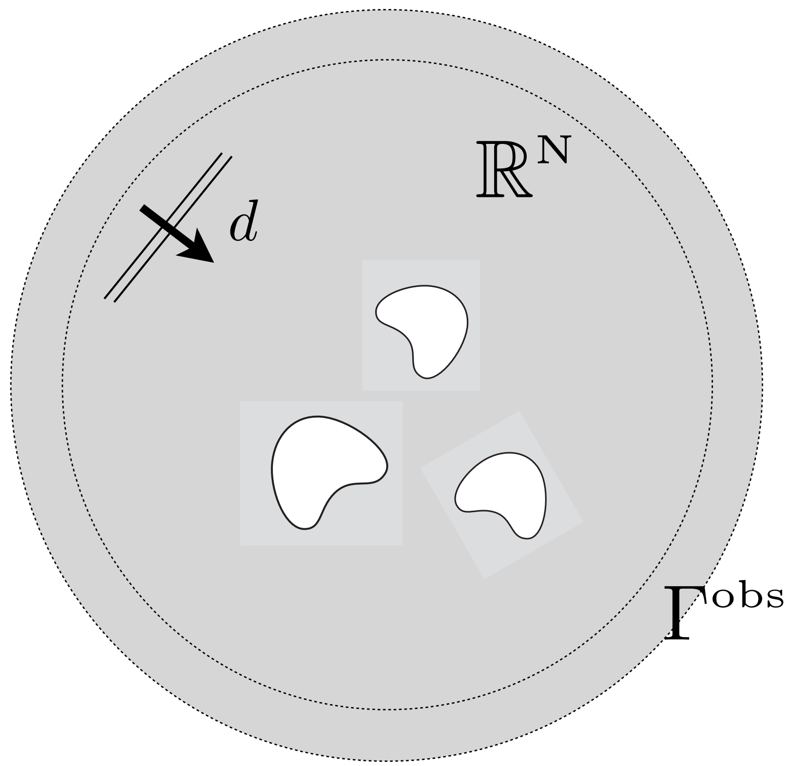
Small

2004-
Guzina, Bonnet, Gallego, Carpio, Rapun, Chikichev, Bellis, Dominguez, Nimitz, Yuan, Malcolm, ...

Extended



TS



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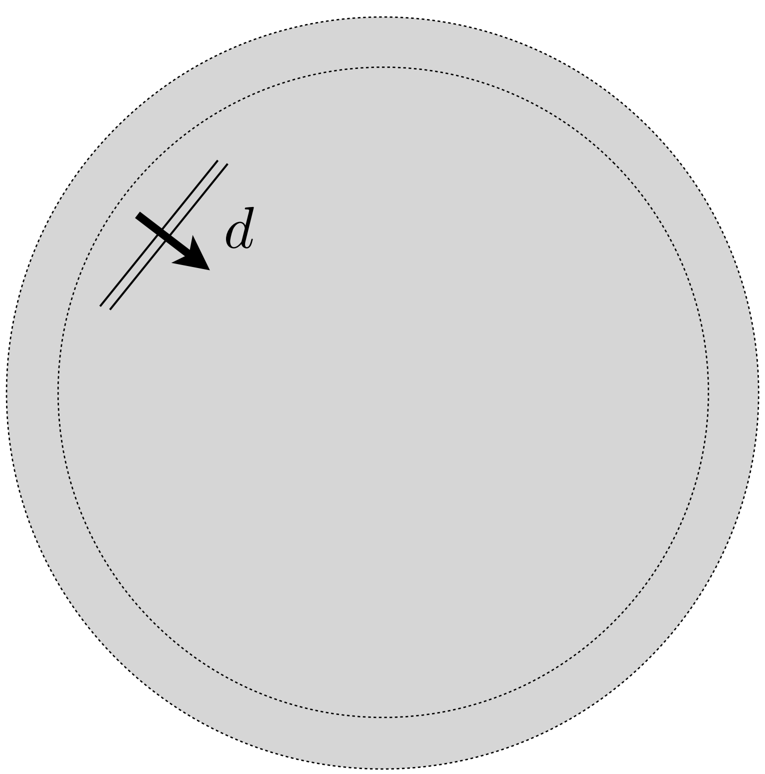
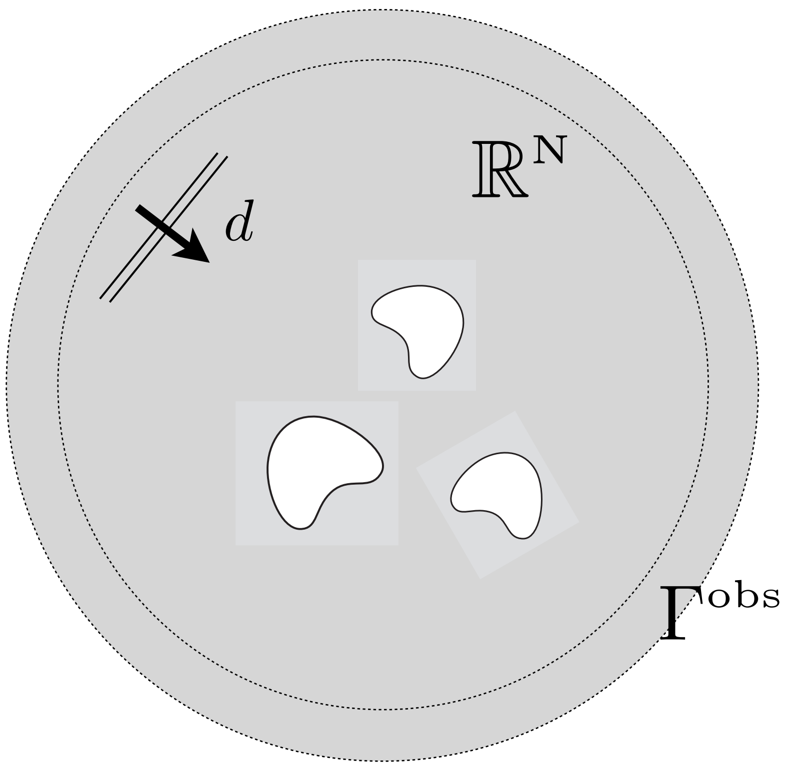


Cost functional

$$J(\emptyset)$$

TS



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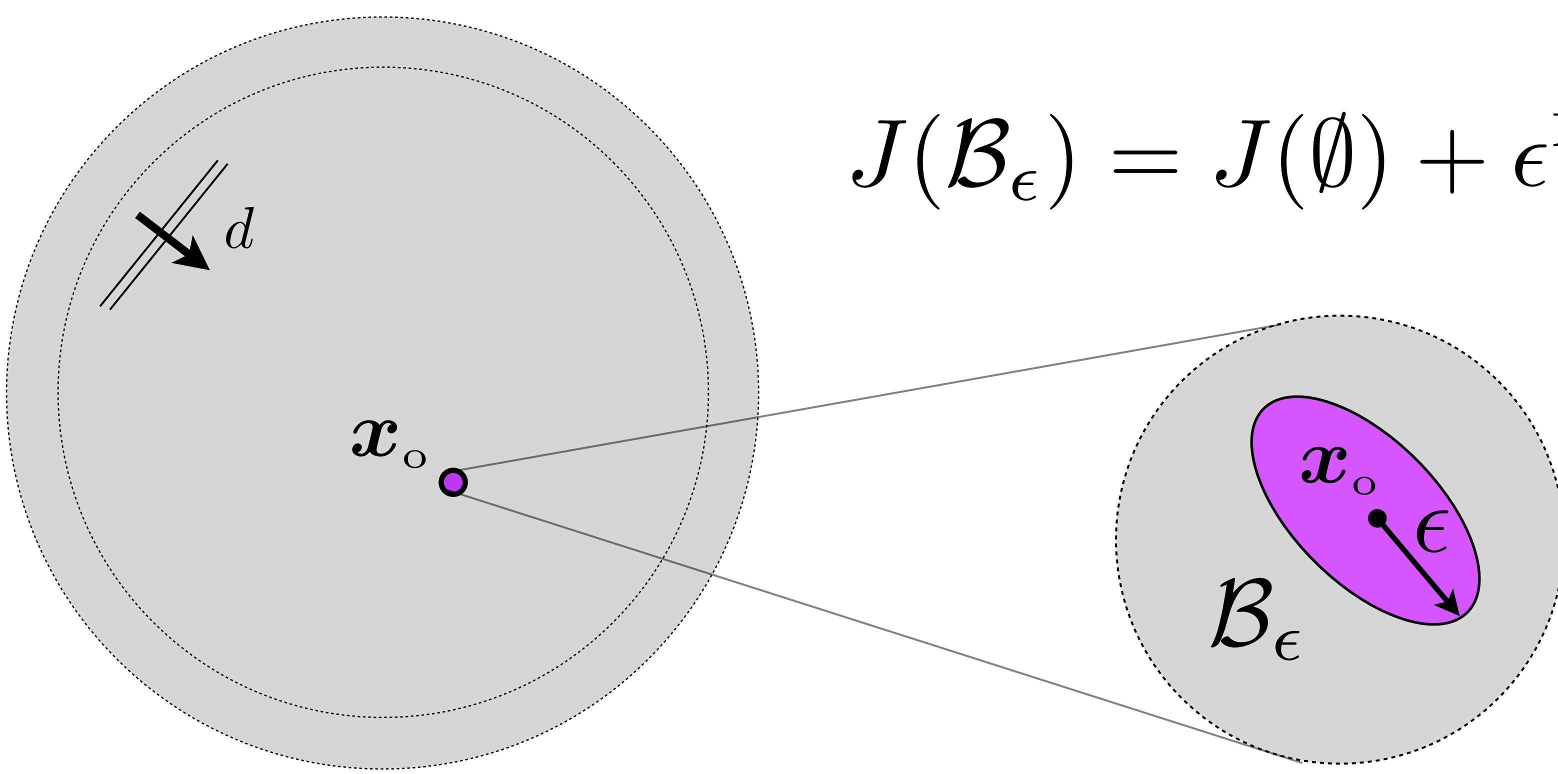
Extended

$$J(\mathcal{B}_\epsilon) = J(\emptyset) + \epsilon^{\mathrm{N}}\,|\mathcal{B}|\,\mathsf{T}(\boldsymbol{x}_\circ) + o(\epsilon^{\mathrm{N}})$$



TS

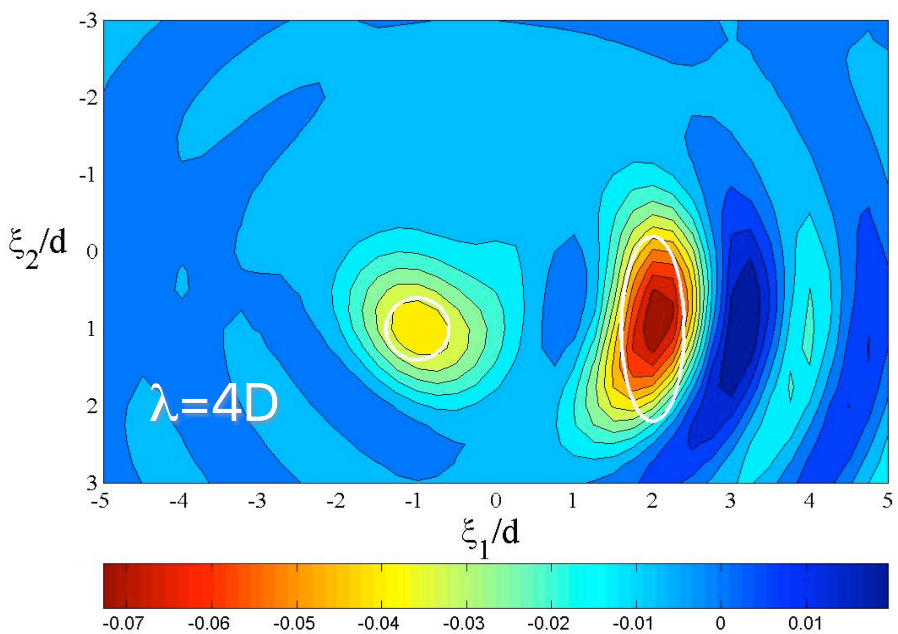
Helmholtz, penetrable

$$\mathsf{T}(\boldsymbol{x}_\circ) \;=\; (1-\beta)\nabla u_{\rm i}\cdot\boldsymbol{A}\cdot\nabla u_{\rm a} \;-\; (1-\eta)\,k^2\,u_{\rm i}\,u_{\rm a}$$

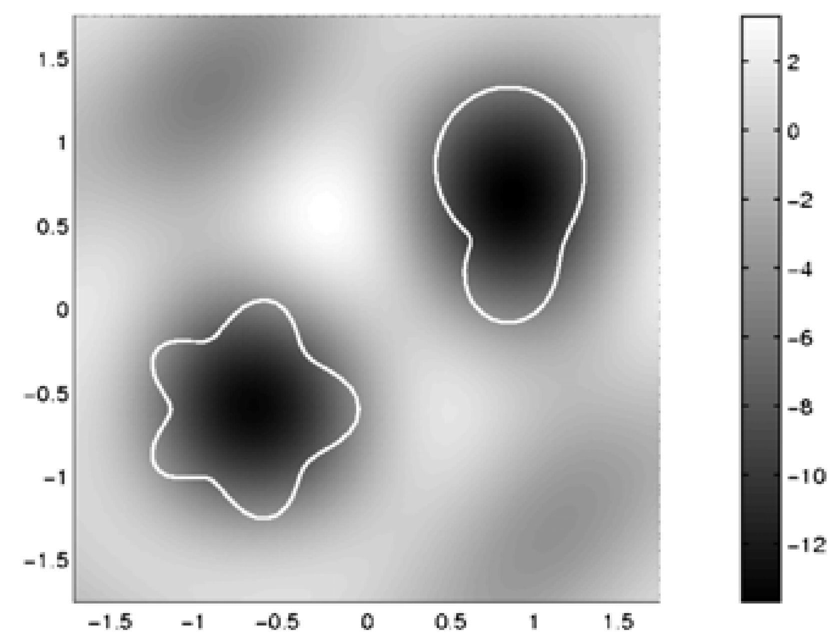
TS

Helmholtz, penetrable

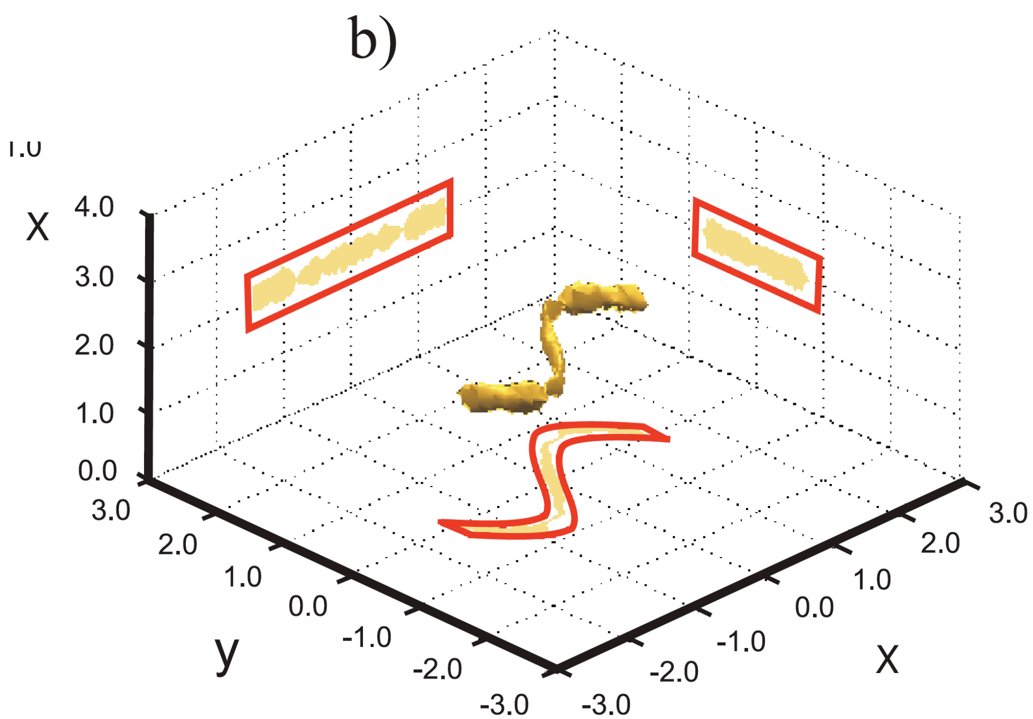
$$\mathrm{T}(\boldsymbol{x}_\circ) = (1-\beta)\nabla u_\mathrm{i}\cdot\boldsymbol{A}\cdot\nabla u_\mathrm{a} - (1-\eta)\,k^2\,u_\mathrm{i}\,u_\mathrm{a}$$



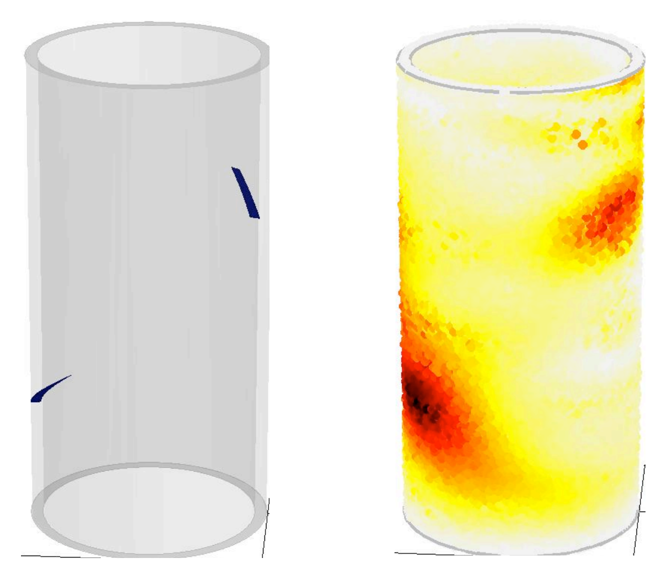
BG & Bonnet (2004) *JMPS*



Carpio & Rapun (2008) *IP*

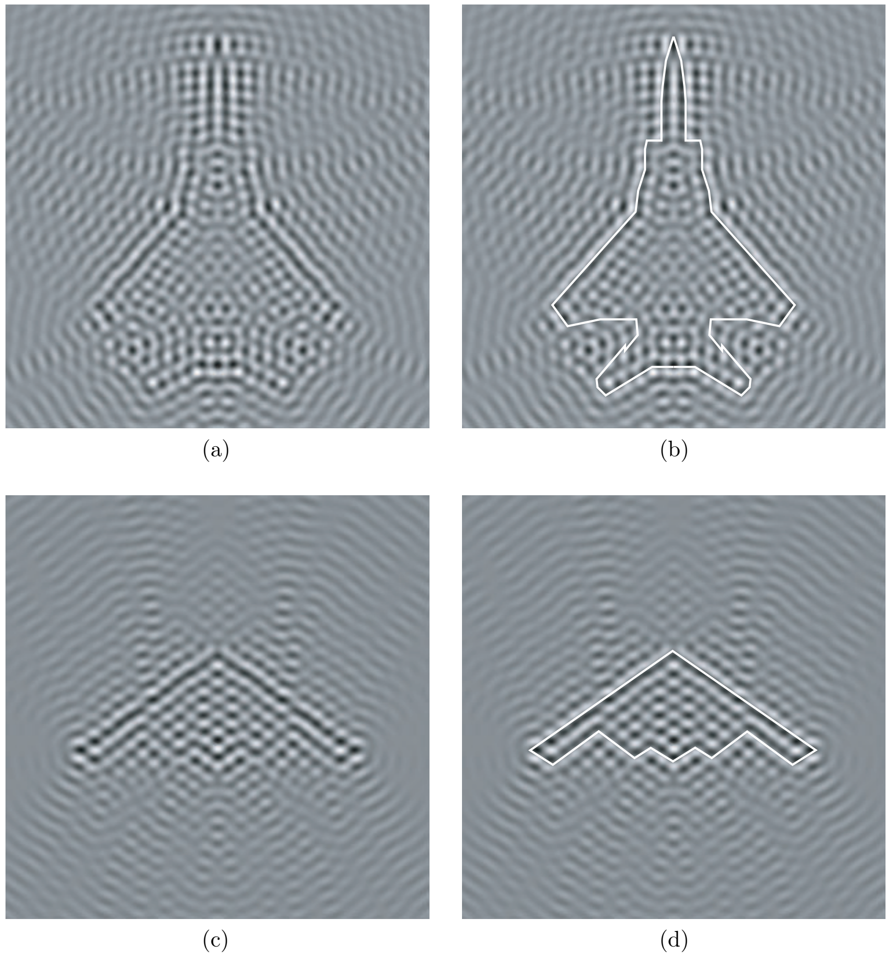


Chikichev & BG (2008) *CMAME*



Bellis & & Bonnet (2013) *CMAME*

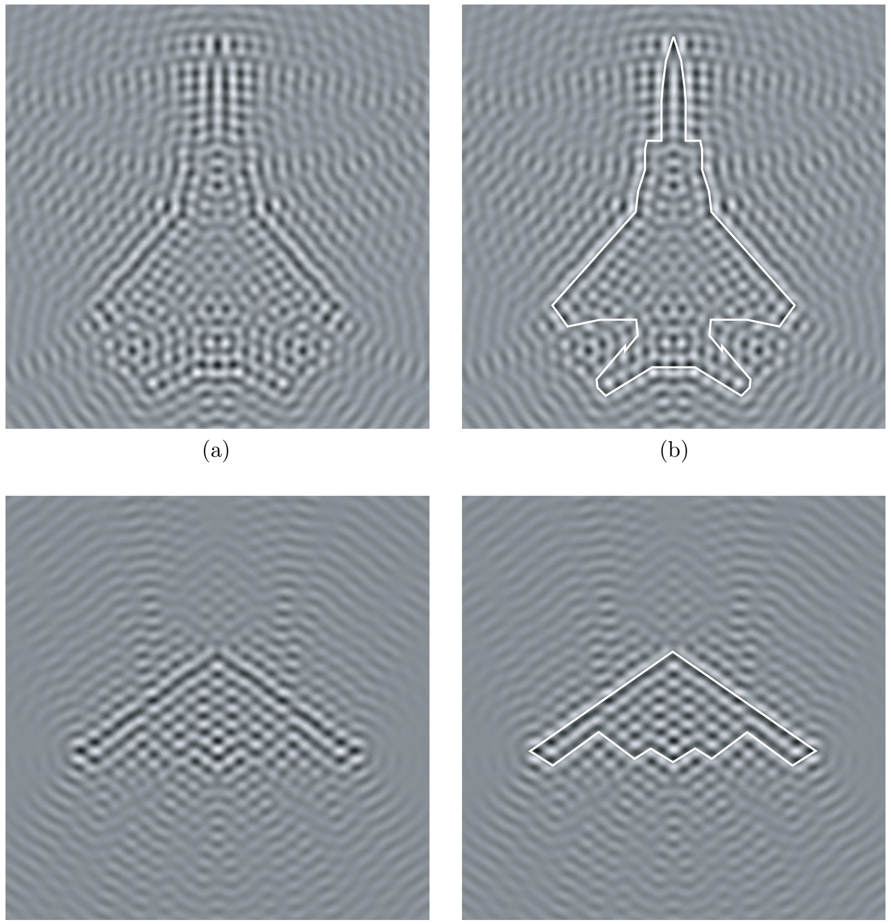
High-frequency TS



Feijoo (2004)
Inverse Problems

scalar, 2D

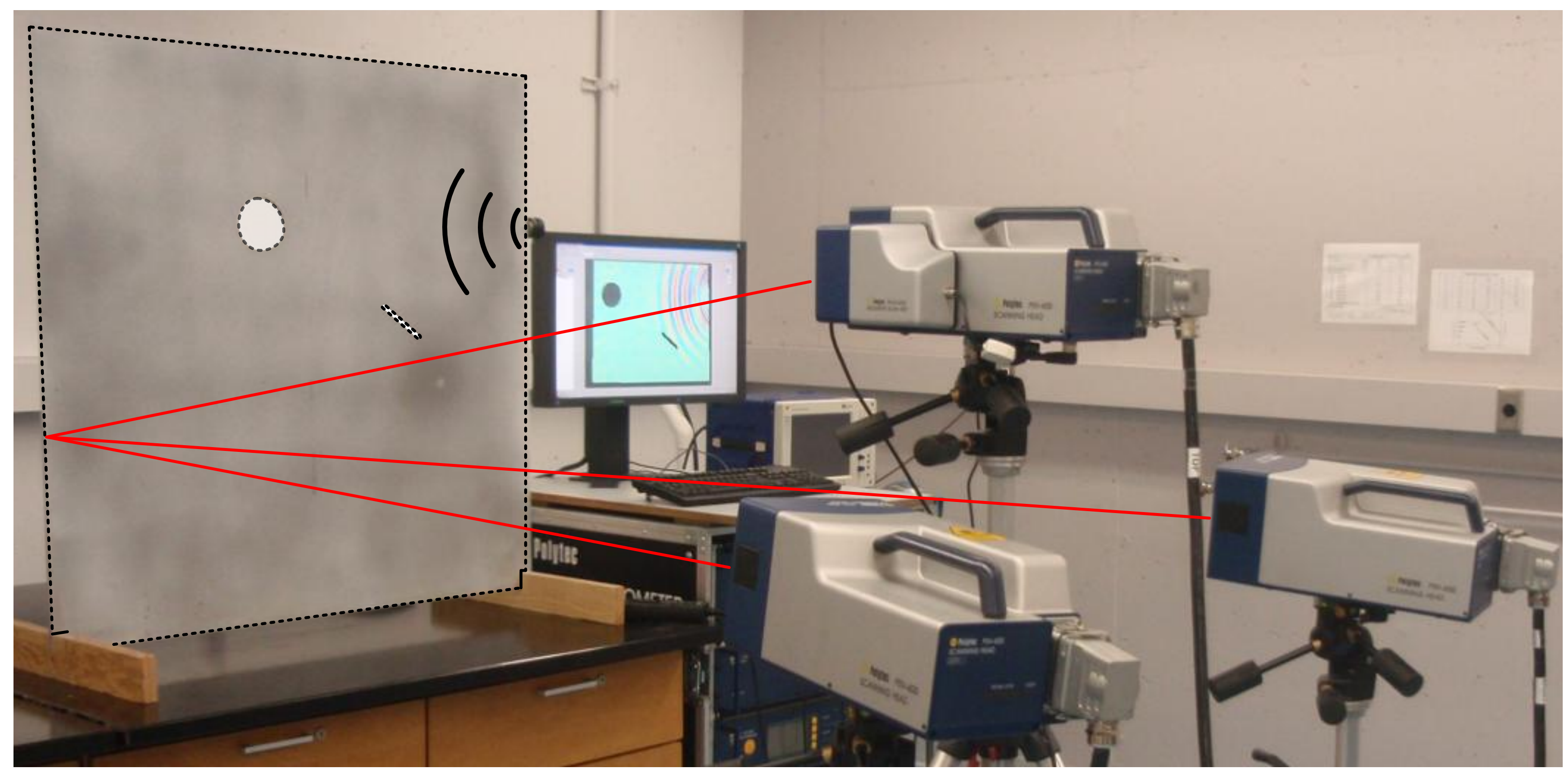
High-frequency TS



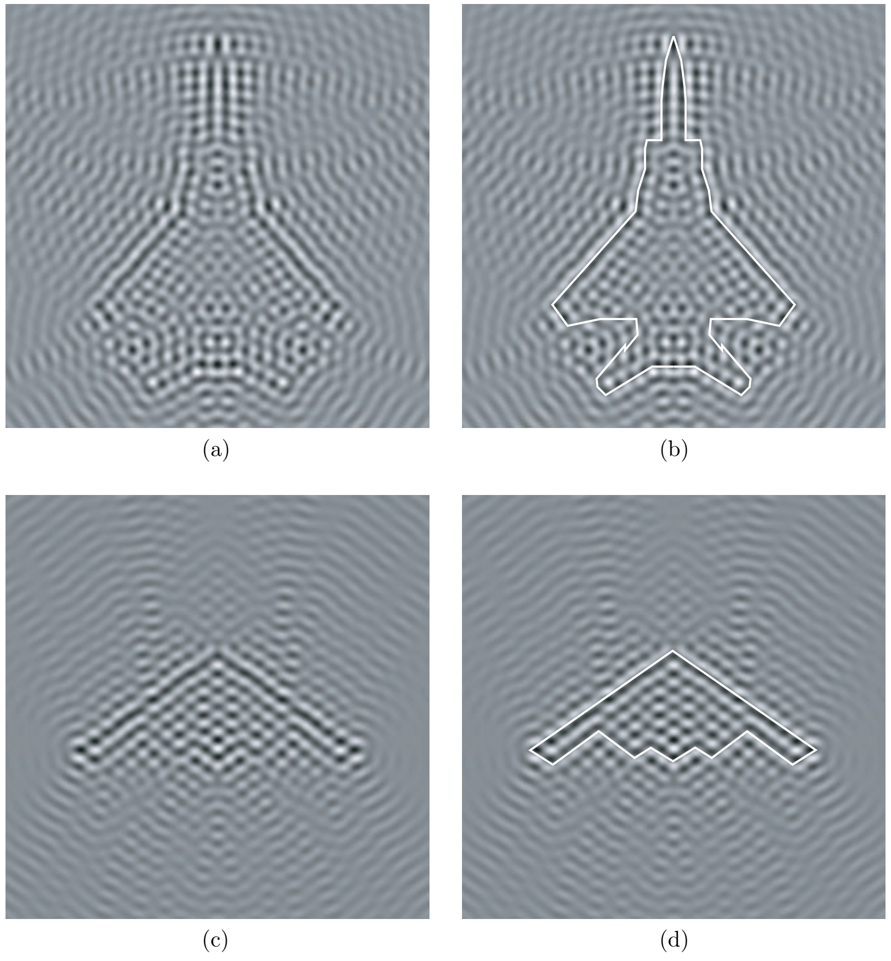
(a) (b) (c) (d)

Feijoo (2004)
Inverse Problems
scalar, 2D

Tokmashev, Tixier & BG (2013) elastodynamic, 2D



High-frequency TS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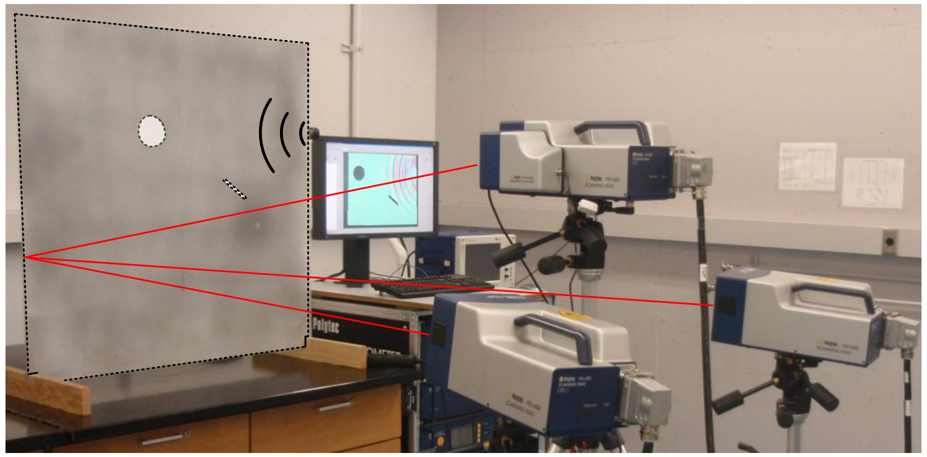
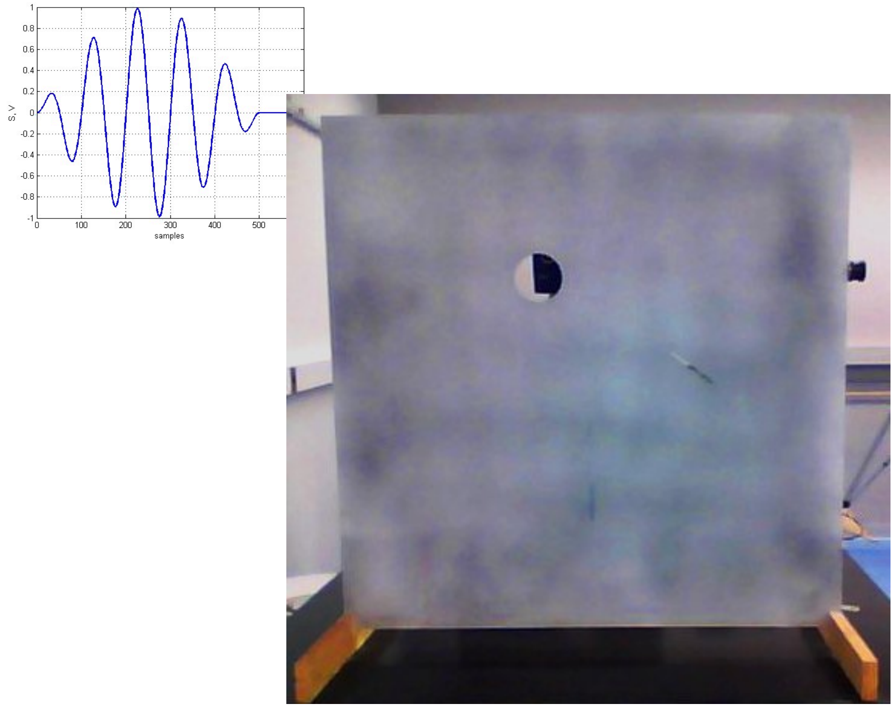
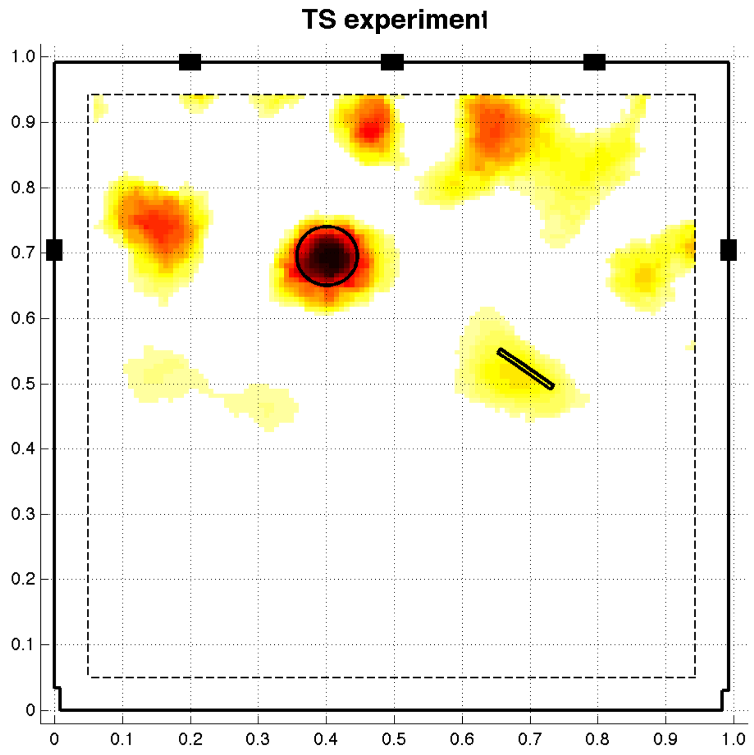
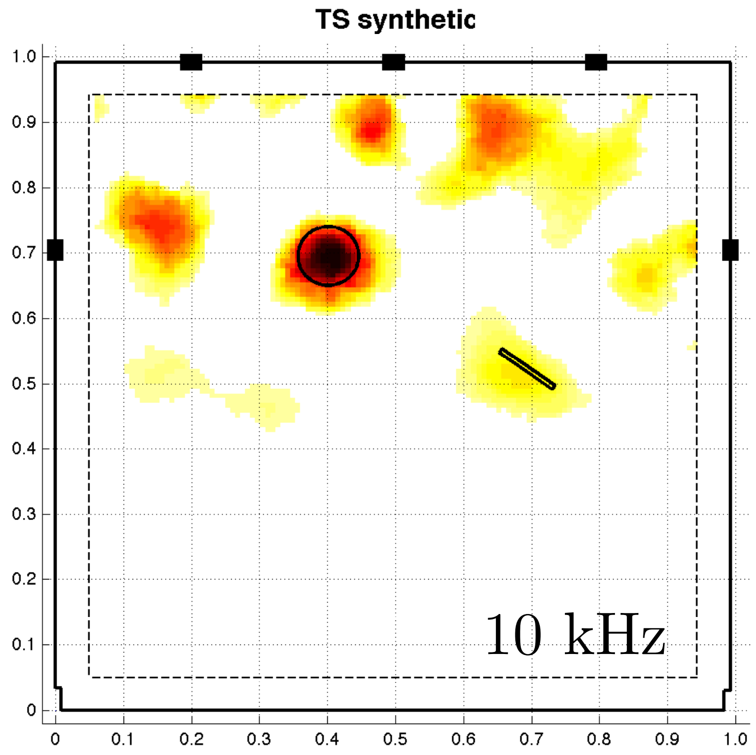
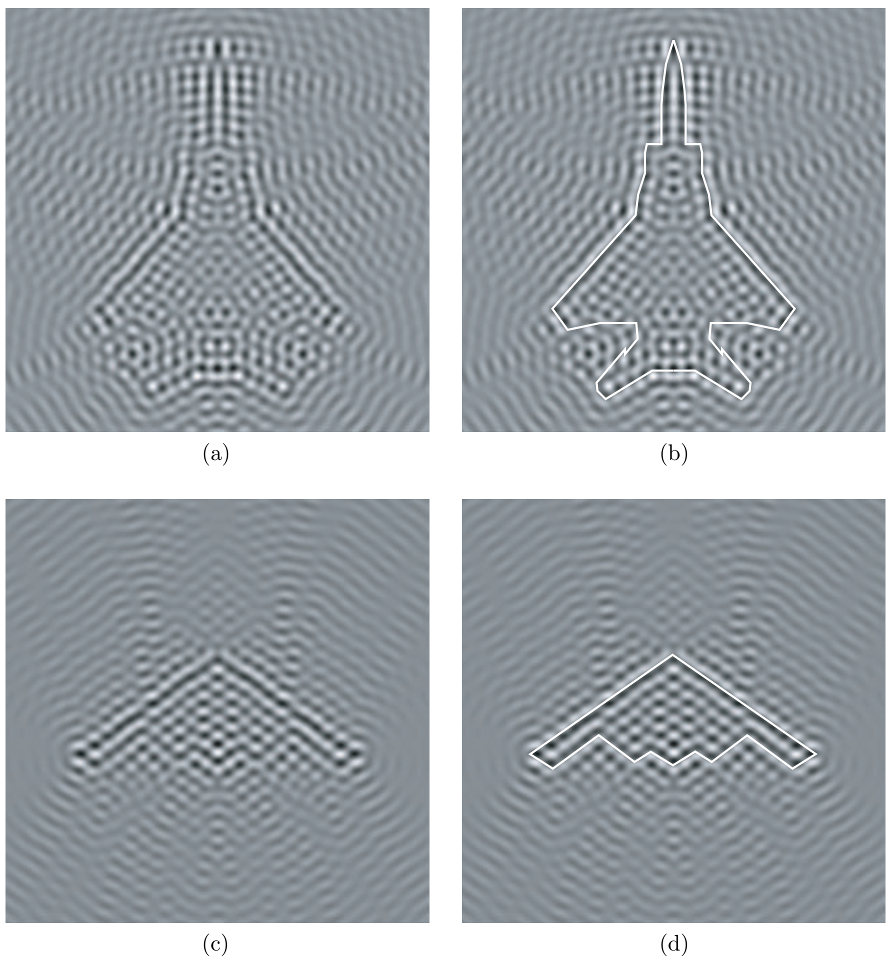








High-frequency TS



Feijoo (2004)
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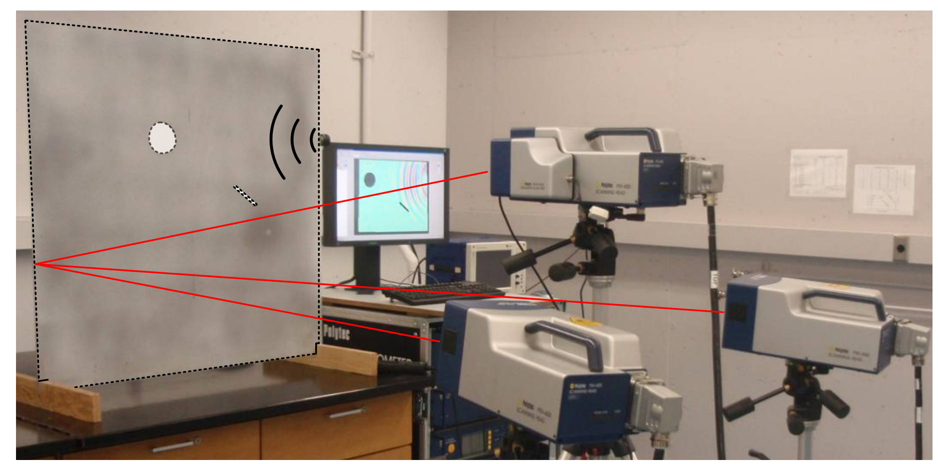
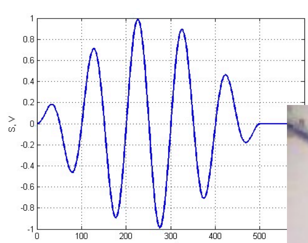
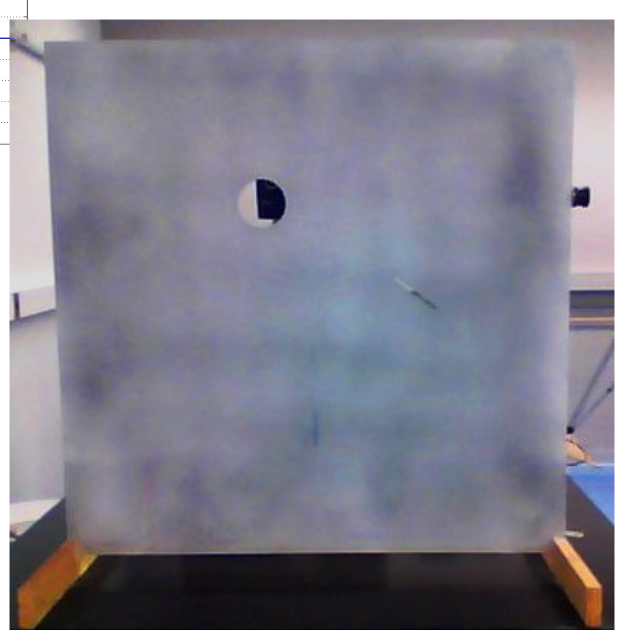
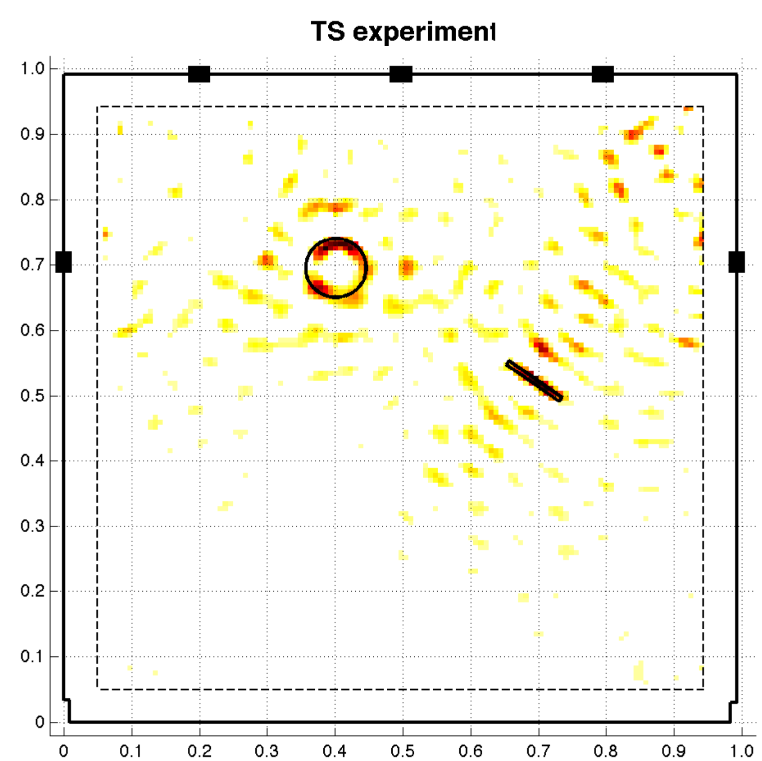
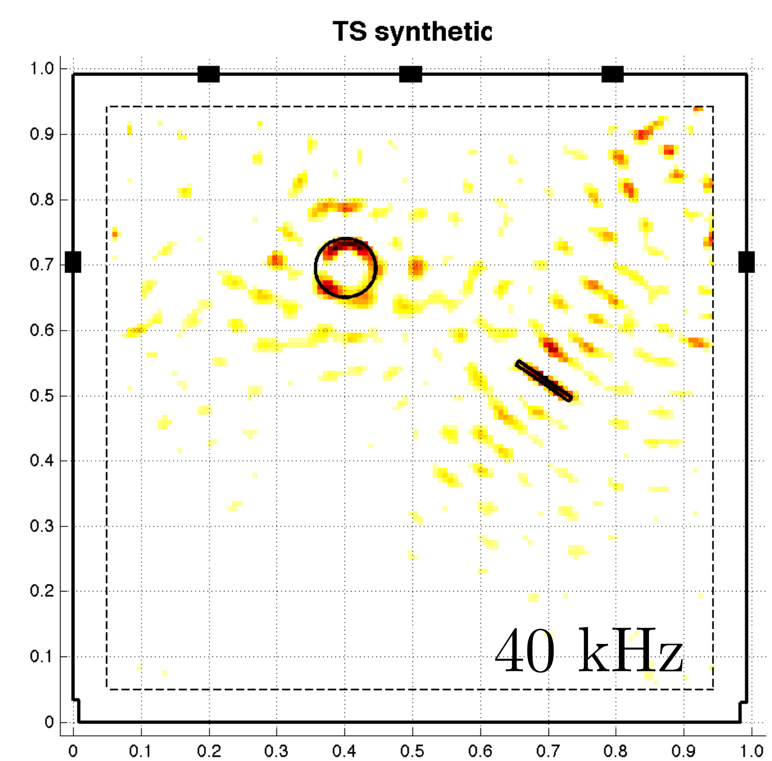




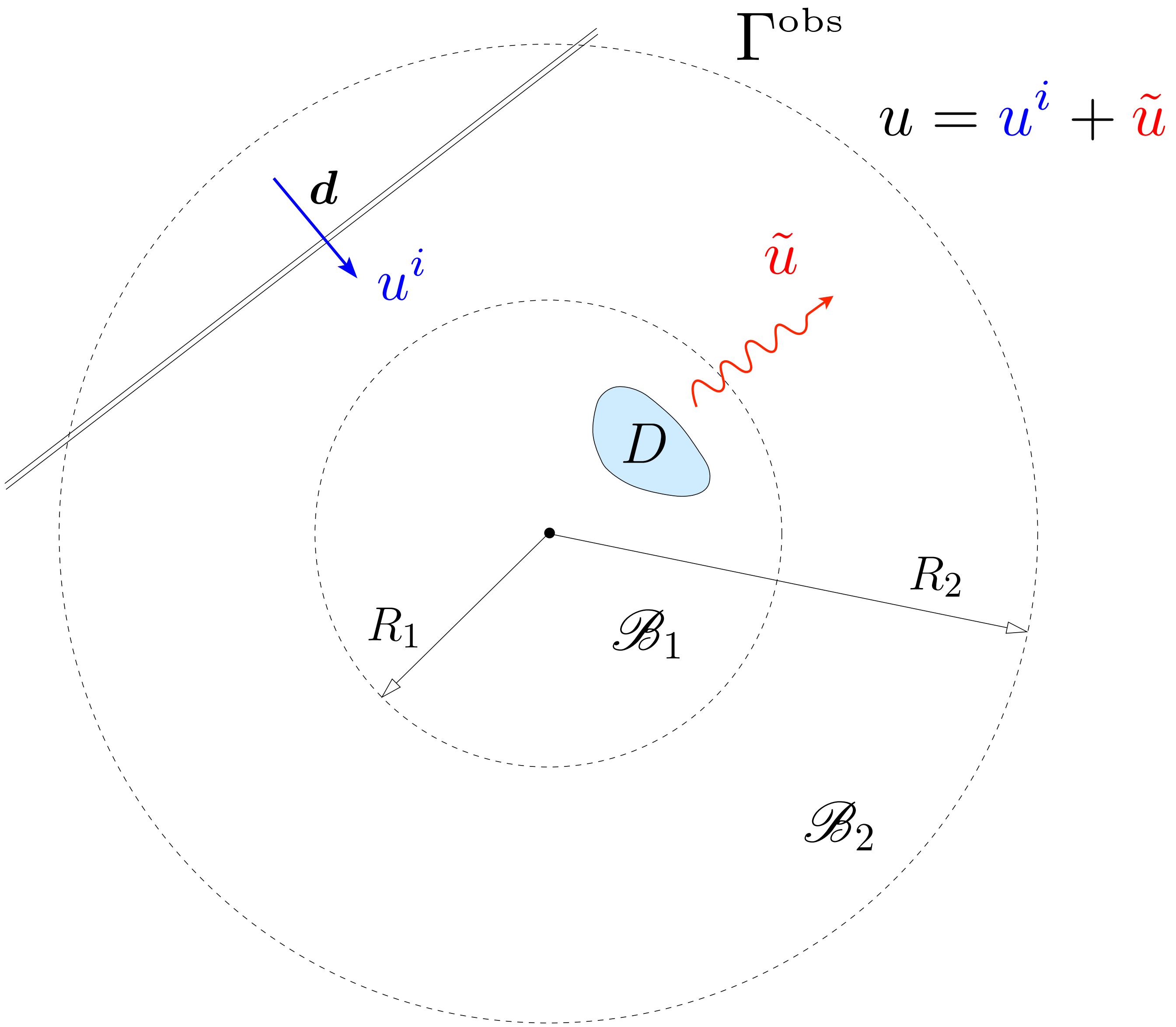






Setup

Acoustic (scalar) in $\mathbb{R}^3$

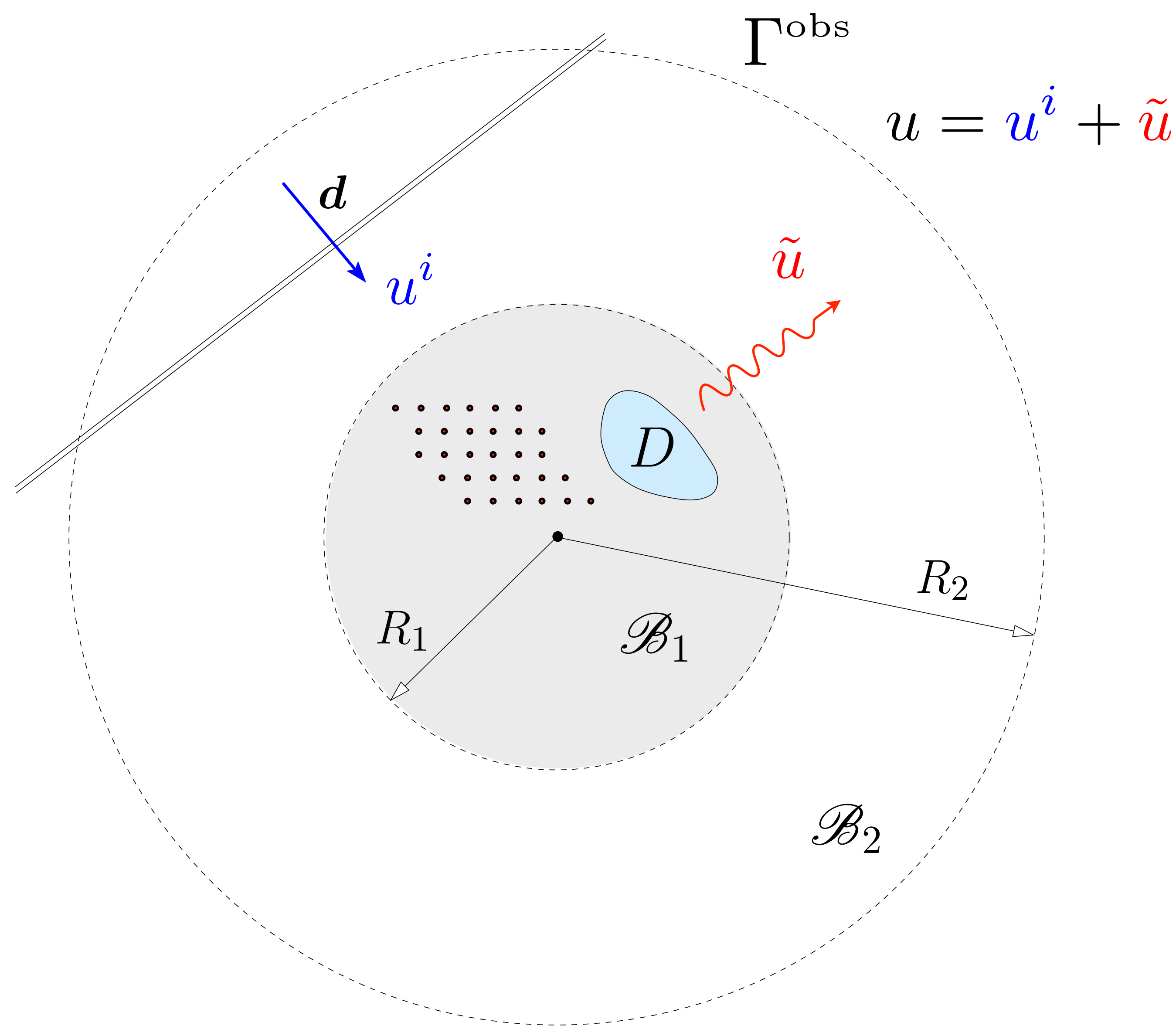


Setup

Acoustic (scalar) in $\mathbb{R}^3$

Dimensional platform: ρ, c, R_1

$$R_1 = 1, \qquad R_2 = \frac{1}{\alpha}$$



Setup

Acoustic (scalar) in $\mathbb{R}^3$

Dimensional platform: ρ, c, R_1

$$R_1 = 1, \qquad R_2 = \frac{1}{\alpha}$$

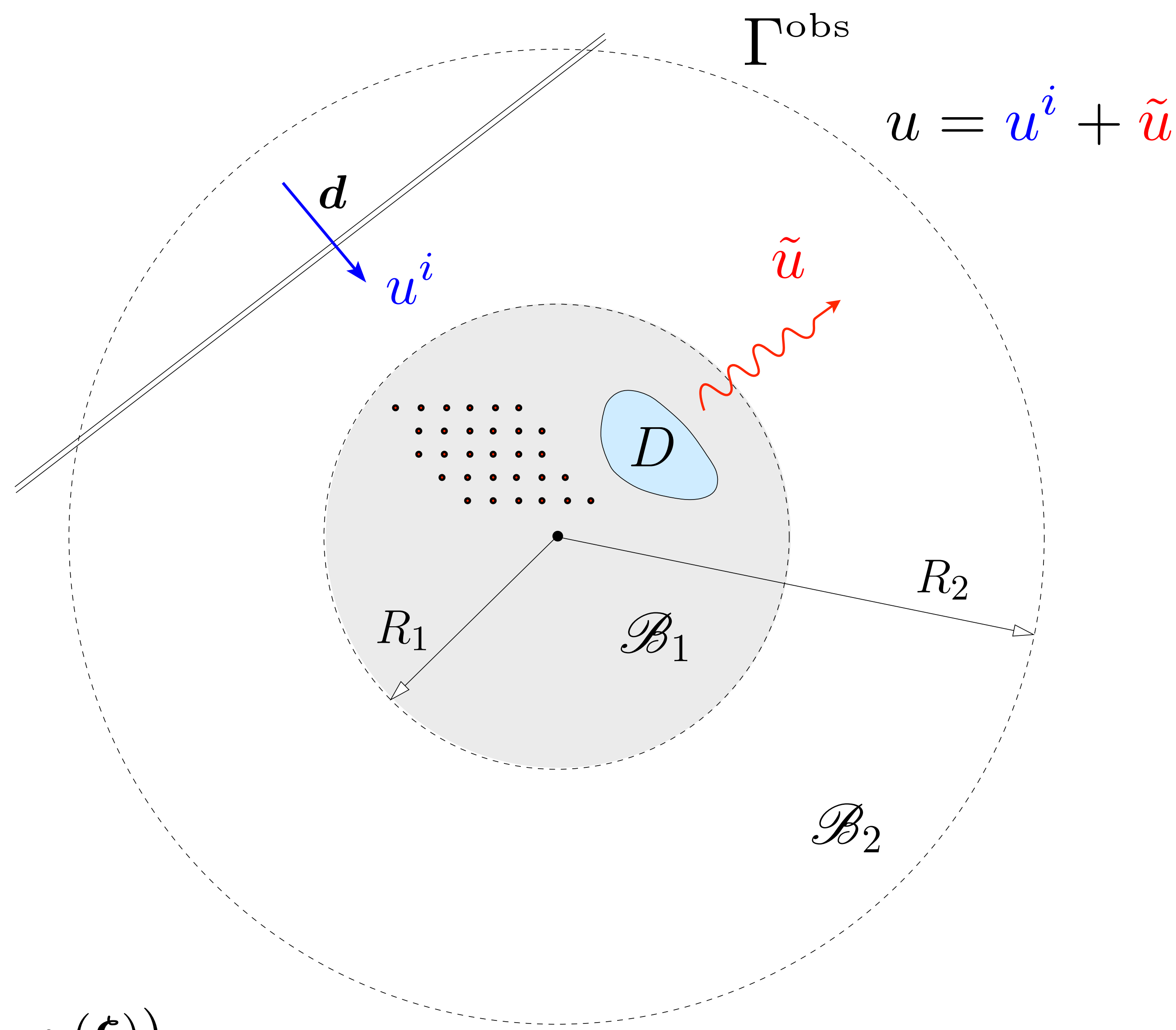
Cost

$$J(\mathscr{D}, \beta, \gamma) := \int_{\Gamma^{\mathrm{obs}}} \varphi\left(v(\boldsymbol{\xi}), u(\boldsymbol{\xi}), \boldsymbol{\xi}\right) \, \mathrm{d}\Gamma_{\xi}$$

$\rho/\rho^\star$ $\quad c/c^\star$

LS

$$\varphi(v(\boldsymbol{\xi}), u(\boldsymbol{\xi}), \boldsymbol{\xi}) = \tfrac{1}{2} \overline{\left(v(\boldsymbol{\xi}) - u(\boldsymbol{\xi})\right)} \left(v(\boldsymbol{\xi}) - u(\boldsymbol{\xi})\right)$$



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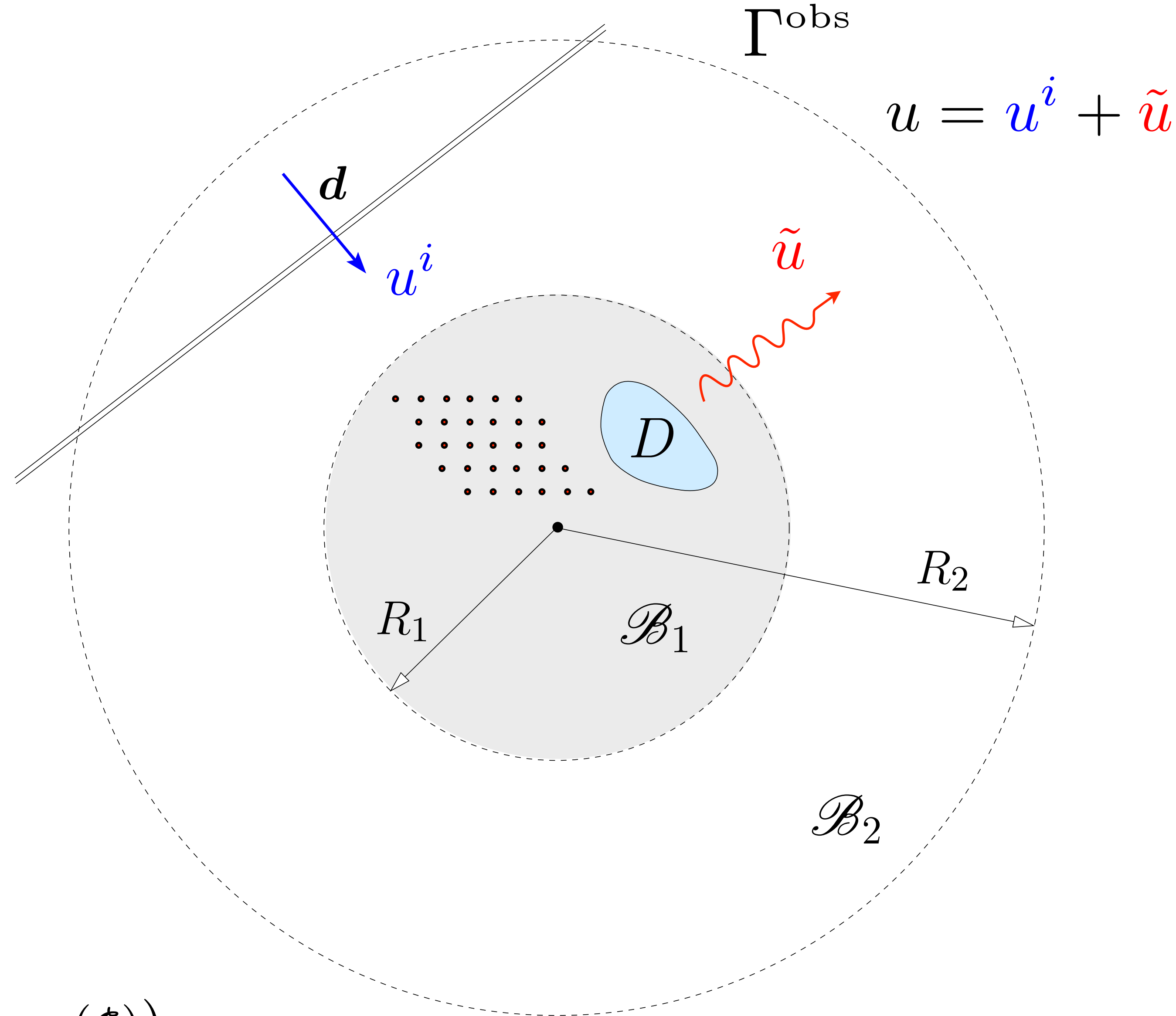
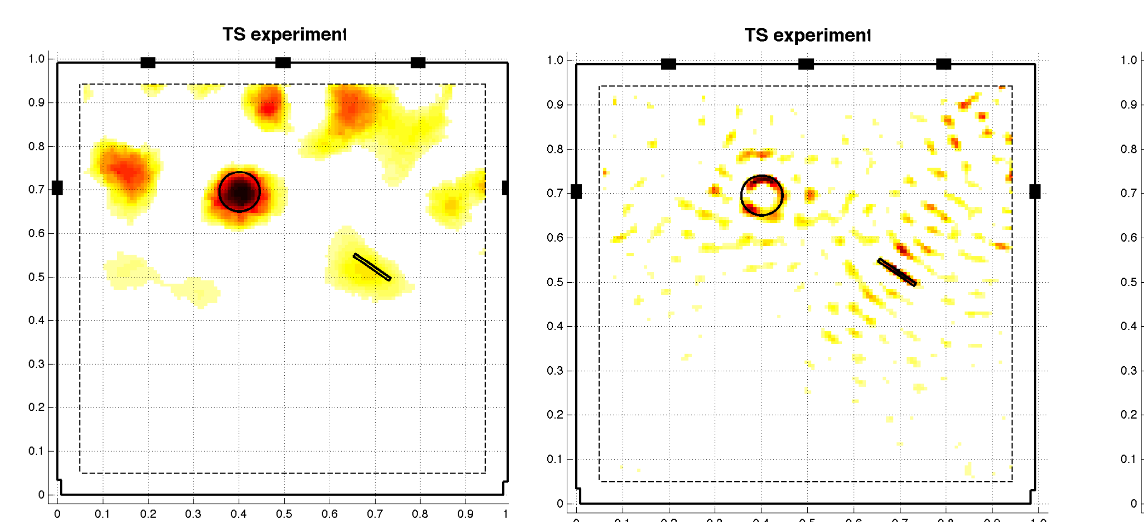


LS

$$\varphi(v(\boldsymbol{\xi}), u(\boldsymbol{\xi}), \boldsymbol{\xi}) \;=\; \tfrac{1}{2}\, \overline{\left(v(\boldsymbol{\xi}) - u(\boldsymbol{\xi})\right)}\, \left(v(\boldsymbol{\xi}) - u(\boldsymbol{\xi})\right)$$

Topological sensitivity

$$J(\mathcal{B}_\epsilon) = J(\emptyset) \;+\; \epsilon^3 |\mathcal{B}|\, \mathsf{T}(\boldsymbol{x}_\circ) + o(\epsilon^3)$$



Bellis, Bonnet, Cakoni (2013), *IP*

Topological sensitivity

$$\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = \int_{\Gamma^{\text{obs}}} \text{Re}\left(\frac{\partial\varphi}{\partial v}\big(u^i(\boldsymbol{\xi}),u(\boldsymbol{\xi}),\boldsymbol{\xi}\big) \Big[(1-\beta)\,\nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot\nabla G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k) - (1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^\circ)\,G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k)\Big]\right) \text{d}\Gamma_\xi$$

$$-\bar{\tilde{u}} = \overline{u^i - u}$$

$$\widetilde{v}_\epsilon$$

Topological sensitivity

$$\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) \;=\; \int_{\Gamma^{\text{obs}}} \text{Re}\left(\frac{\partial\varphi}{\partial v}(u^i(\boldsymbol{\xi}),u(\boldsymbol{\xi}),\boldsymbol{\xi}) \left[(1-\beta)\,\nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot\nabla G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k) - (1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^\circ)\,G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k) \right] \right) \text{d}\Gamma_\xi$$

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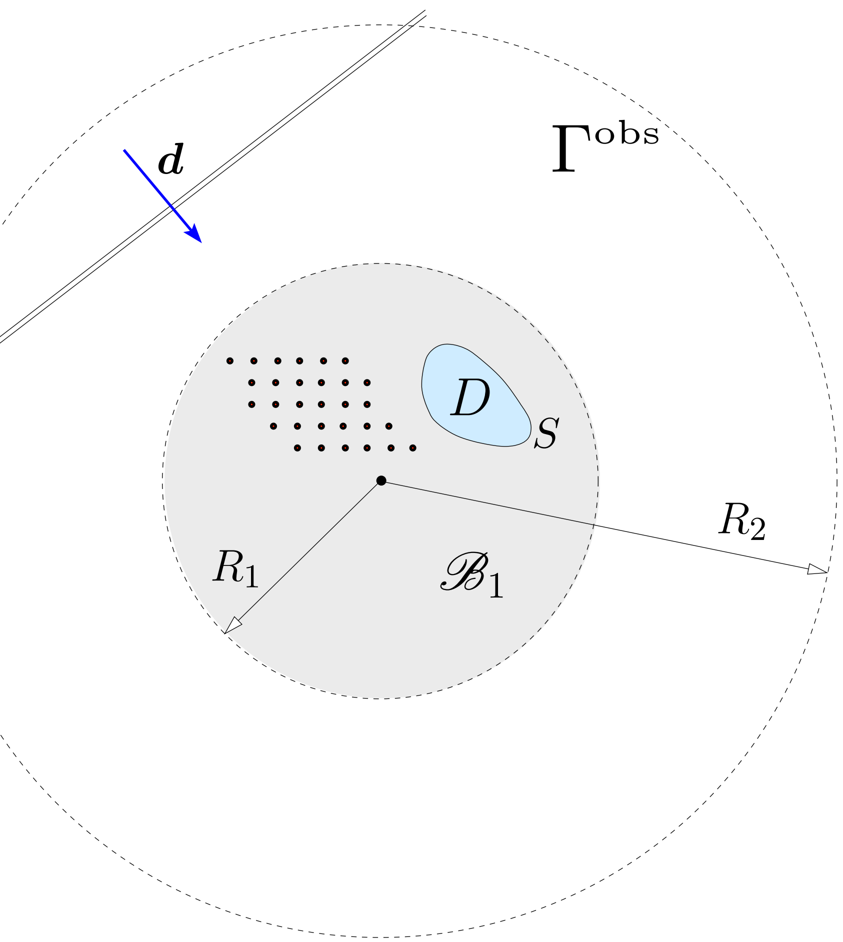

Integral representation of the true scattered field

$$u^i(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \;=\; \int_S \Big(u_{,n}(\boldsymbol{\zeta})\,G(\boldsymbol{\zeta},\boldsymbol{\xi},k) \;-\; u(\boldsymbol{\zeta})\,\boldsymbol{n}(\boldsymbol{\zeta})\cdot\nabla G(\boldsymbol{\zeta},\boldsymbol{\xi},k) \Big)\text{d}S_\zeta$$

Topological sensitivity

$$\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = \int_{\Gamma^{\text{obs}}} \text{Re}\left(\frac{\partial \varphi}{\partial v}\big(u^i(\boldsymbol{\xi}), u(\boldsymbol{\xi}), \boldsymbol{\xi}\big) \Big[(1-\beta)\, \nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot\nabla G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k) - (1-\beta\gamma^2)\, k^2\, u^i(\boldsymbol{x}^\circ)\, G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k) \Big] \right) \text{d}\Gamma_\xi$$

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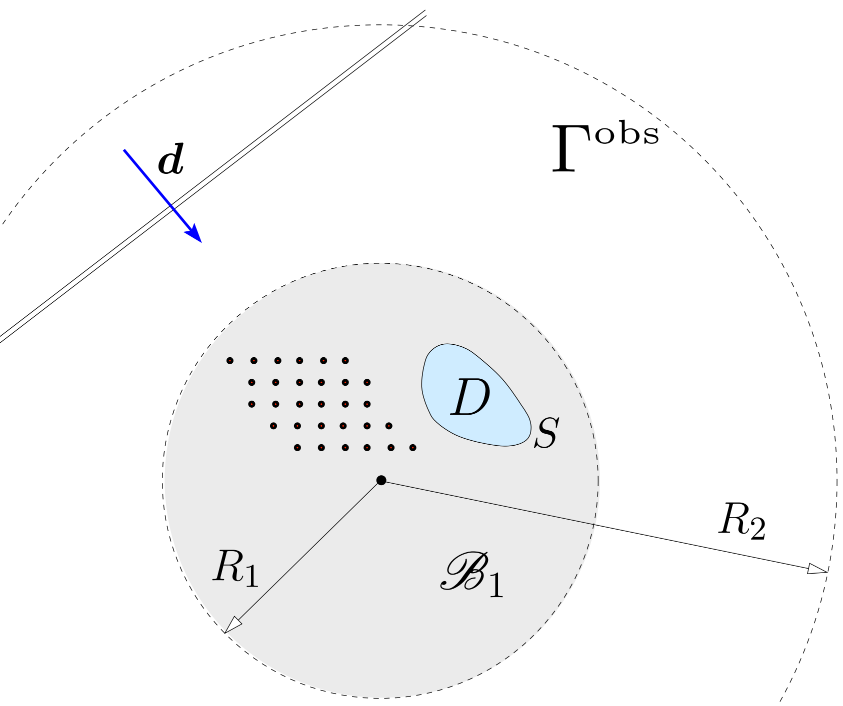

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$$\begin{aligned}
\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = -\,\text{Re}\Big\{ &(1-\beta)\, \nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot \Big[\int_S \overline{u}_{,n}(\boldsymbol{\zeta}) \int_{\Gamma^{\text{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\, \nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\, \text{d}\Gamma_\xi\, \text{d}S_\zeta \\
&+ \int_S \overline{u}(\boldsymbol{\zeta})\, \boldsymbol{n}(\boldsymbol{\zeta})\cdot \int_{\Gamma^{\text{obs}}} \nabla\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k) \otimes \nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\, \text{d}\Gamma_\xi\, \text{d}S_\zeta \Big] \\
&+ (1-\beta\gamma^2)\, k^2\, u^i(\boldsymbol{x}^\circ) \Big[\int_S \overline{u}_{,n}(\boldsymbol{\zeta}) \int_{\Gamma^{\text{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\, G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\, \text{d}\Gamma_\xi\, \text{d}S_\zeta \\
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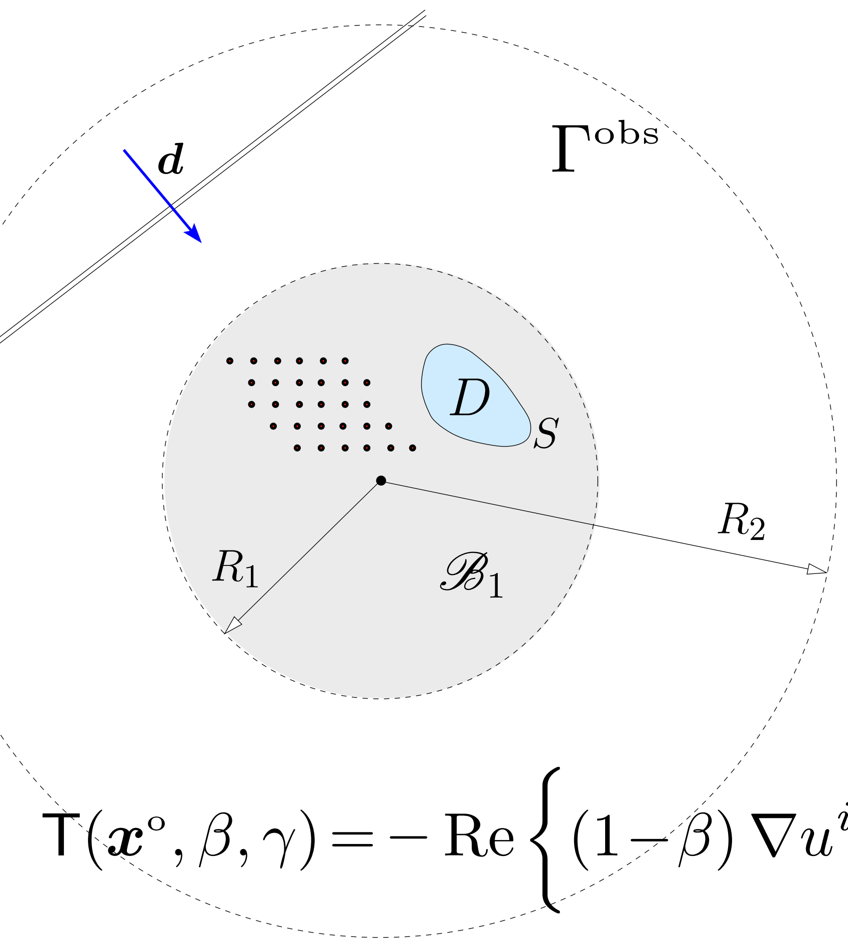
\end{aligned}$$

Topological sensitivity

$$\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = \int_{\Gamma^{\text{obs}}} \text{Re}\left(\frac{\partial\varphi}{\partial v}(u^i(\boldsymbol{\xi}),u(\boldsymbol{\xi}),\boldsymbol{\xi}) \left[(1-\beta)\,\nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot\nabla G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k) - (1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^\circ)\,G(\boldsymbol{x}^\circ,\boldsymbol{\xi},k)\right]\right)\text{d}\Gamma_\xi$$

$$-\tilde{\bar{u}} = \overline{u^i - u}$$

$$\widetilde{v}_\epsilon$$



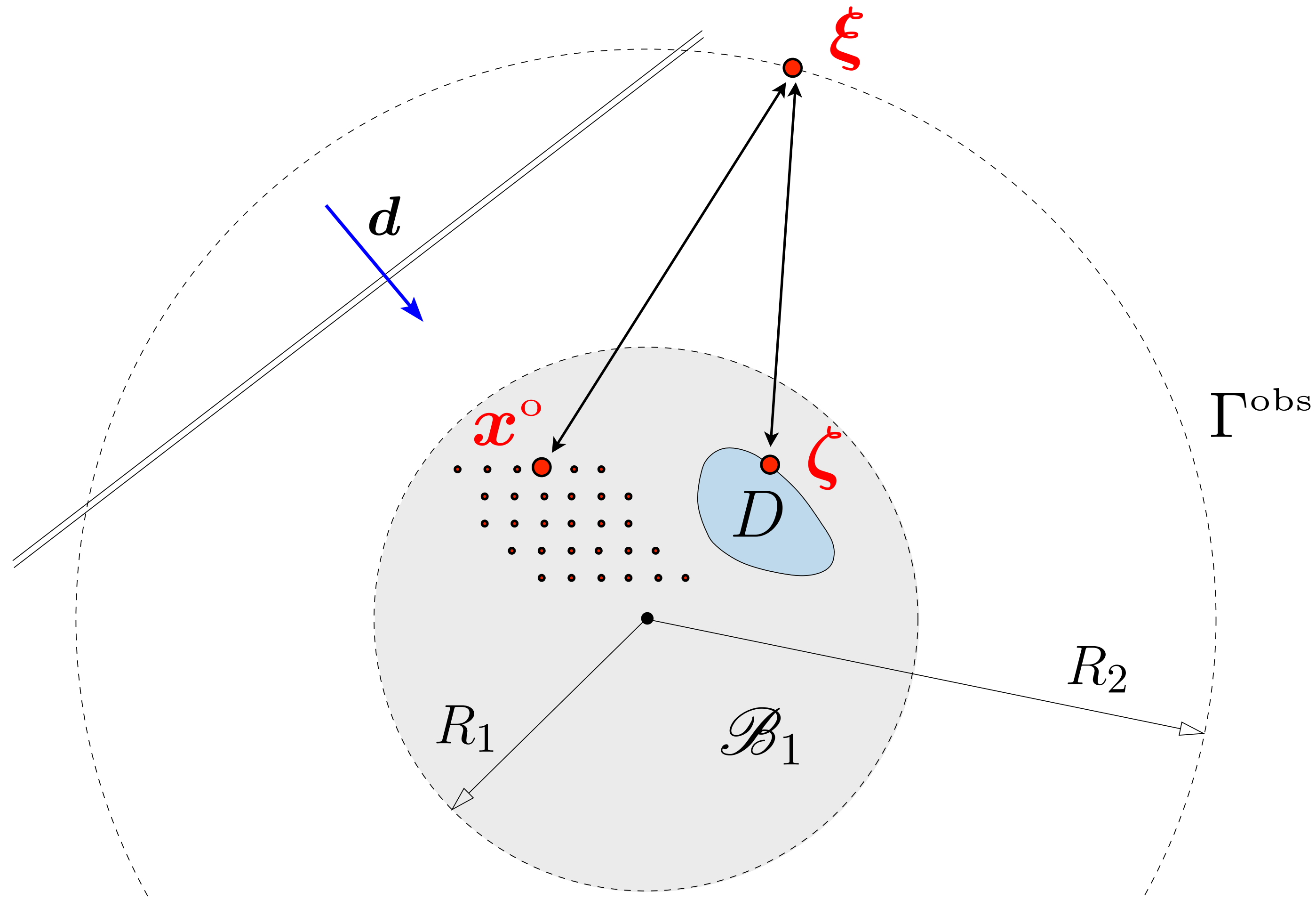
Integral representation of the true scattered field

$$u^i(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) = \int_S \Big(u_{,n}(\boldsymbol{\zeta})\,G(\boldsymbol{\zeta},\boldsymbol{\xi},k) - u(\boldsymbol{\zeta})\,\boldsymbol{n}(\boldsymbol{\zeta})\cdot\nabla G(\boldsymbol{\zeta},\boldsymbol{\xi},k)\Big)\text{d}S_\zeta$$

$$\begin{aligned}\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = -\,\text{Re}\Big\{ &(1-\beta)\,\nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot\Big[\int_S \overline{u}_{,n}(\boldsymbol{\zeta}) \int_{\Gamma^{\text{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,\nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\text{d}\Gamma_\xi\,\text{d}S_\zeta \\ &+ \int_S \overline{u}(\boldsymbol{\zeta})\,\boldsymbol{n}(\boldsymbol{\zeta})\cdot\int_{\Gamma^{\text{obs}}} \nabla\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\otimes\nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\text{d}\Gamma_\xi\,\text{d}S_\zeta\Big] \\ &+ (1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^\circ)\Big[\int_S \overline{u}_{,n}(\boldsymbol{\zeta}) \int_{\Gamma^{\text{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\text{d}\Gamma_\xi\,\text{d}S_\zeta \\ &+ \int_S \overline{u}(\boldsymbol{\zeta})\,\boldsymbol{n}(\boldsymbol{\zeta})\cdot\int_{\Gamma^{\text{obs}}} \nabla\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\text{d}\Gamma_\xi\,\text{d}S_\zeta\Big]\Big\},\end{aligned}$$

Multipole expansion

$$\alpha = \frac{R_1}{R_2}$$



Multipole expansion

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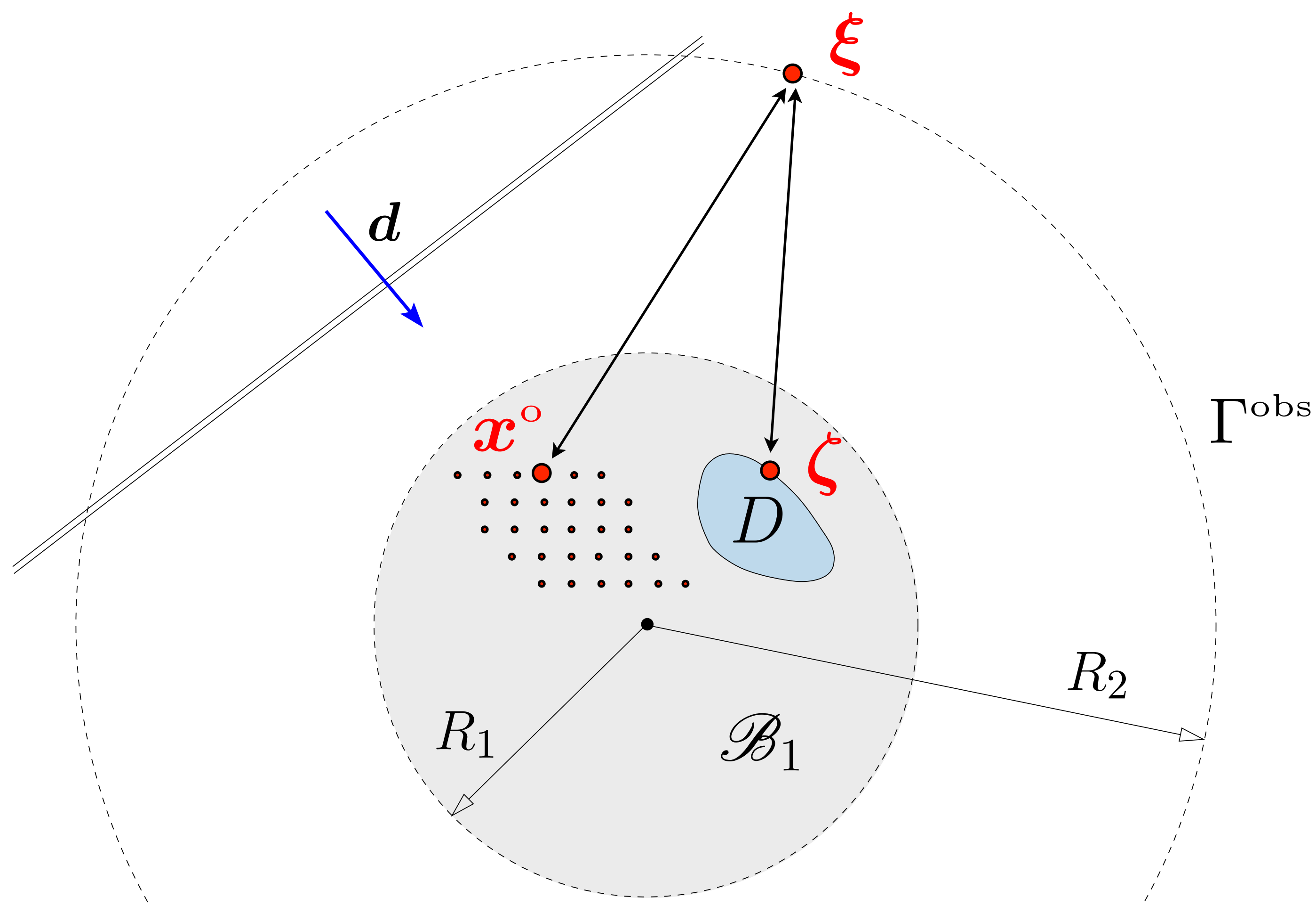

$$\begin{aligned}\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = -\,\mathrm{Re}\Bigg\{ &(1-\beta)\,\nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot\Bigg[\int_S \overline{u}_{,n}(\boldsymbol{\zeta})\int_{\Gamma^{\text{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,\nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi\,\mathrm{d}S_\zeta \\ &+ \int_S \overline{u}(\boldsymbol{\zeta})\,\boldsymbol{n}(\boldsymbol{\zeta})\cdot\int_{\Gamma^{\text{obs}}} \nabla\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\otimes\nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi\,\mathrm{d}S_\zeta\Bigg] \\ &+ (1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^\circ)\Bigg[\int_S \overline{u}_{,n}(\boldsymbol{\zeta})\int_{\Gamma^{\text{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi\,\mathrm{d}S_\zeta \\ &+ \int_S \overline{u}(\boldsymbol{\zeta})\,\boldsymbol{n}(\boldsymbol{\zeta})\cdot\int_{\Gamma^{\text{obs}}} \nabla\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi\,\mathrm{d}S_\zeta\Bigg]\Bigg\},\end{aligned}$$

Multipole expansion

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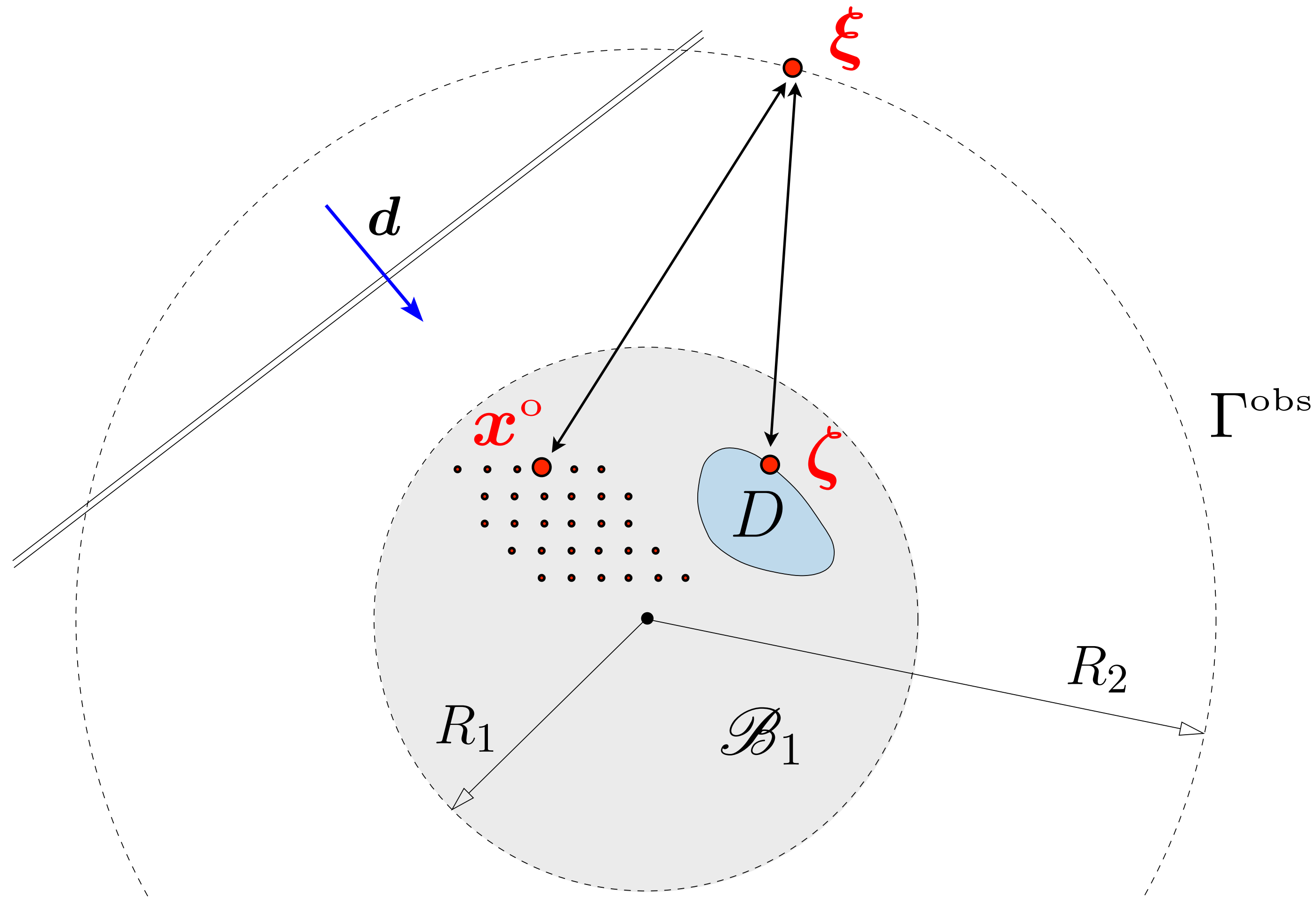

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&+ (1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^\circ)\Big[\int_S \overline{u}_{,n}(\boldsymbol{\zeta}) \int_{\Gamma^{\mathrm{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi\,\mathrm{d}S_\zeta \\
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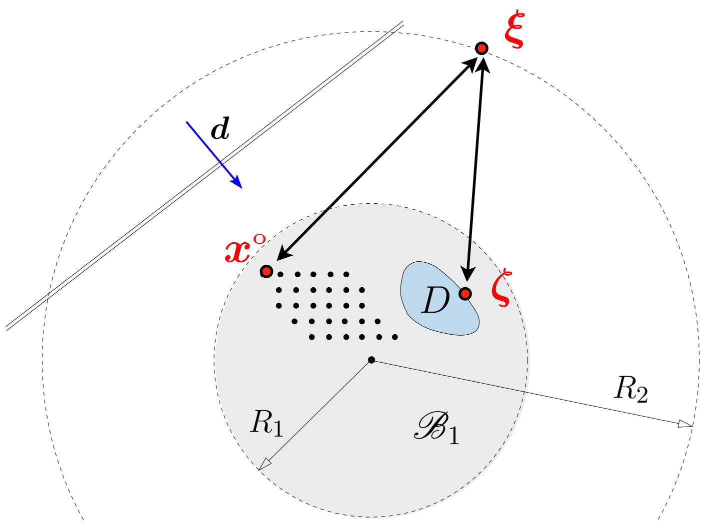
\end{aligned}$$

Helmholtz-Kirchhoff



$$\int_{\Gamma^{\text{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\, G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi \overset{\alpha^2}{=} -\frac{1}{k}\,\mathrm{Im}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big)$$

e.g. Blackstock (2000), Garnier & Papanicolaou (2009)

$$\int_{\Gamma^{\text{obs}}} \nabla\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\, G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi \overset{\alpha^2}{=} \Big[\mathrm{Re}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big) + \frac{1}{kr}\,\mathrm{Im}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big)\Big]\,(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})$$

$$\int_{\Gamma^{\text{obs}}} \nabla\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\otimes\nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi \overset{\alpha^2}{=}$$

$$\frac{1}{r}\Big[3\,\mathrm{Re}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big) + \Big(\frac{3}{kr}-kr\Big)\mathrm{Im}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big)\Big](\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})\otimes(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})$$

$$-\frac{1}{r}\Big[\mathrm{Re}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big) + \frac{1}{kr}\mathrm{Im}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big)\Big]\,\boldsymbol{I},$$

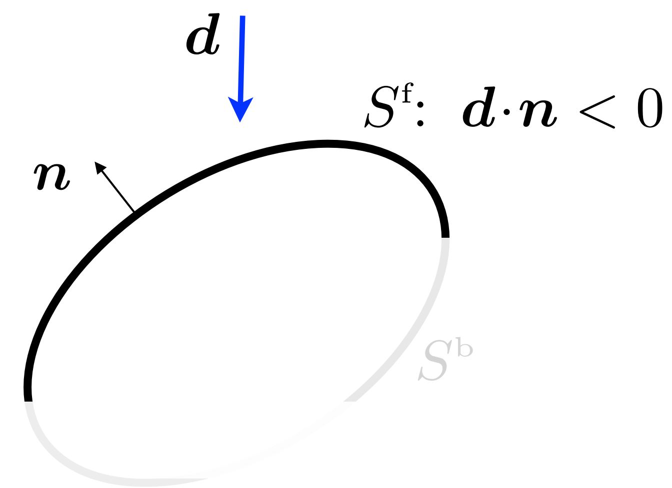
$$\mathrm{Re}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big) = \frac{1}{8\pi r}\big(e^{\mathrm{i}kr}+e^{-\mathrm{i}kr}\big), \qquad \mathrm{Im}\big(G(\boldsymbol{x}^\circ,\boldsymbol{\zeta},k)\big) = \frac{\mathrm{i}}{8\pi r}\big(e^{\mathrm{i}kr}-e^{-\mathrm{i}kr}\big)$$

Dirichlet obstacle, large k

Incident plane wave $u^i = e^{-\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{d}}$

Kirchhoff approximation $kL \gg 1$

$$u = 0 \quad \text{on} \quad S = \partial D, \qquad u_{,n} = \begin{cases} 2\,u^i_{,n} & \text{on} \quad S^{\mathrm{f}} \\ 0 & \text{on} \quad S^{\mathrm{b}} \end{cases}$$



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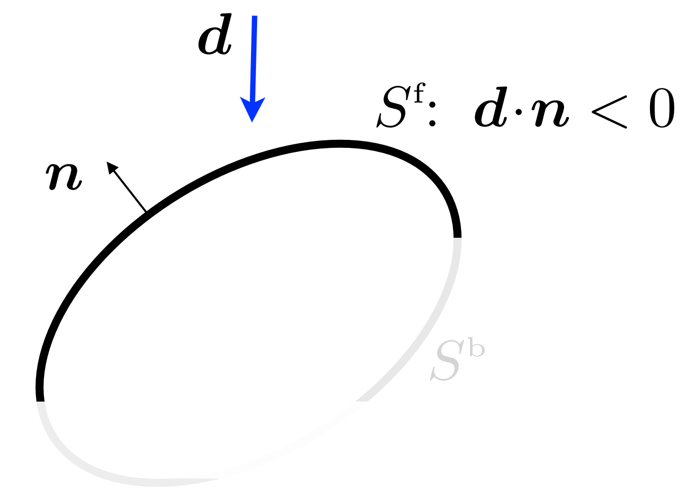

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Sensitivity

ball

$$\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = 2\,\mathrm{Re}\Big\{(1-\beta)\,\nabla u^i(\boldsymbol{x}^\circ)\cdot\boldsymbol{A}\cdot\int_{S^{\mathrm{f}}}\overline{u^i}_{,n}(\boldsymbol{\zeta})\int_{\Gamma^{\mathrm{obs}}}\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,\nabla G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi\,\mathrm{d}S_\zeta$$
$$-\,(1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^\circ)\int_{S^{\mathrm{f}}}\overline{u^i}_{,n}(\boldsymbol{\zeta})\int_{\Gamma^{\mathrm{obs}}}\overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,G(\boldsymbol{\xi},\boldsymbol{x}^\circ,k)\,\mathrm{d}\Gamma_\xi\,\mathrm{d}S_\zeta\Big\}$$

$$\mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma) = 2k^2\,\mathrm{Im}\Big\{\frac{3(1-\beta)}{2+\beta}\,(-\mathrm{i}\,e^{-\mathrm{i}k\boldsymbol{x}^\circ\cdot\boldsymbol{d}})\,J_1 \;-\; (1-\beta\gamma^2)\,(e^{-\mathrm{i}k\boldsymbol{x}^\circ\cdot\boldsymbol{d}})\,J_2\Big\}$$

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Incident plane wave $\quad u^i = e^{-\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{d}}$

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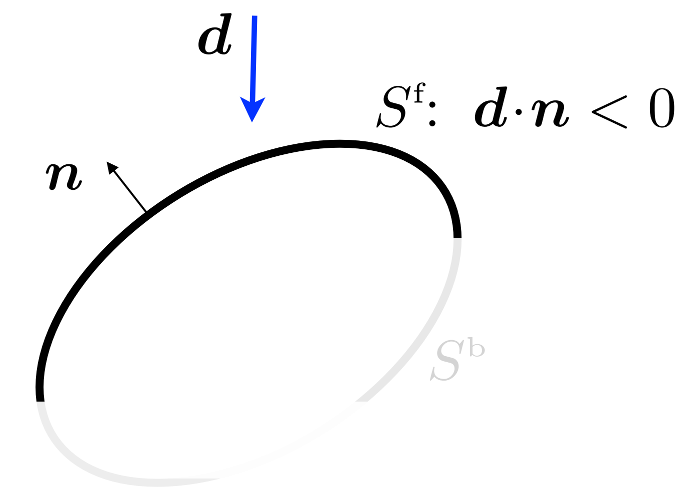

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Sensitivity

ball

$$\mathsf{T}(\boldsymbol{x}^{\circ},\beta,\gamma) = 2\,\mathrm{Re}\Big\{(1-\beta)\,\nabla u^i(\boldsymbol{x}^{\circ})\cdot\boldsymbol{A}\cdot\int_{S^{\mathrm{f}}} \overline{u^i_{,n}}(\boldsymbol{\zeta}) \int_{\Gamma^{\mathrm{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,\nabla G(\boldsymbol{\xi},\boldsymbol{x}^{\circ},k)\,\mathrm{d}\Gamma_{\xi}\,\mathrm{d}S_{\zeta}$$
$$- (1-\beta\gamma^2)\,k^2\,u^i(\boldsymbol{x}^{\circ}) \int_{S^{\mathrm{f}}} \overline{u^i_{,n}}(\boldsymbol{\zeta}) \int_{\Gamma^{\mathrm{obs}}} \overline{G}(\boldsymbol{\xi},\boldsymbol{\zeta},k)\,G(\boldsymbol{\xi},\boldsymbol{x}^{\circ},k)\,\mathrm{d}\Gamma_{\xi}\,\mathrm{d}S_{\zeta}\Big\}$$

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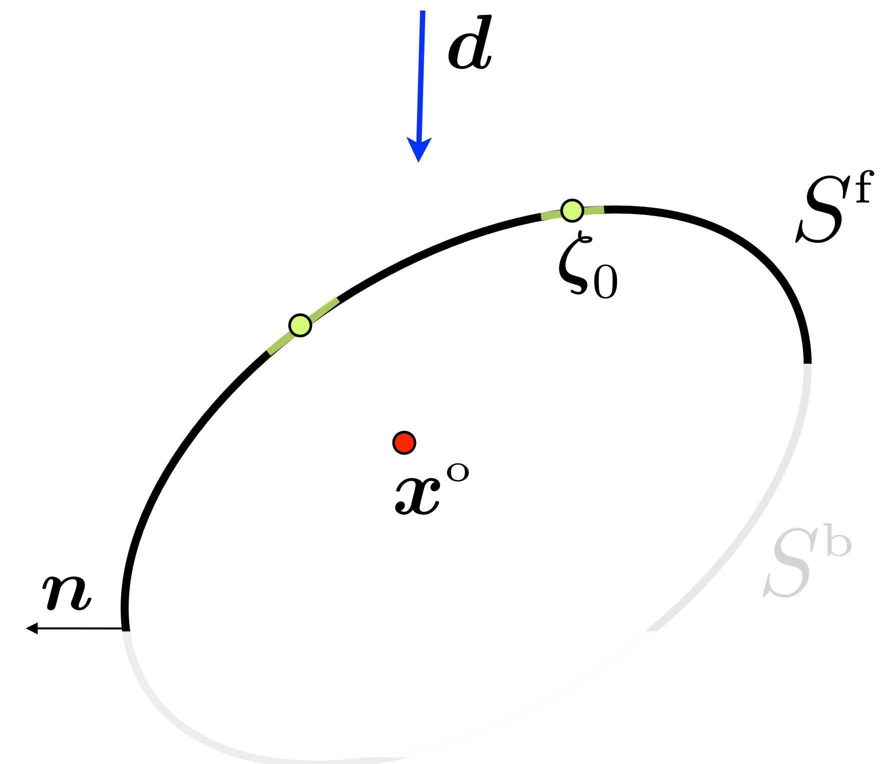
$$J_1 = \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1+\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^{\circ}-\boldsymbol{\zeta})}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_{\zeta} + \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1-\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^{\circ}-\boldsymbol{\zeta})}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_{\zeta}$$

$$J_2 = \mathrm{i}\int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_{\zeta} - \mathrm{i}\int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_{\zeta}, \qquad r = |\boldsymbol{x}^{\circ}-\boldsymbol{\zeta}|, \qquad \boldsymbol{x}^{\circ} \notin S^{\mathrm{f}}$$

Oscillatory integral

"slow" "fast"

$$\mathsf{T}(\boldsymbol{x}^{\circ}) = \int_{S^{\rm f}} f(\zeta)\, e^{\mathrm{i}k\varphi(\zeta)}\, \mathrm{d}\zeta$$

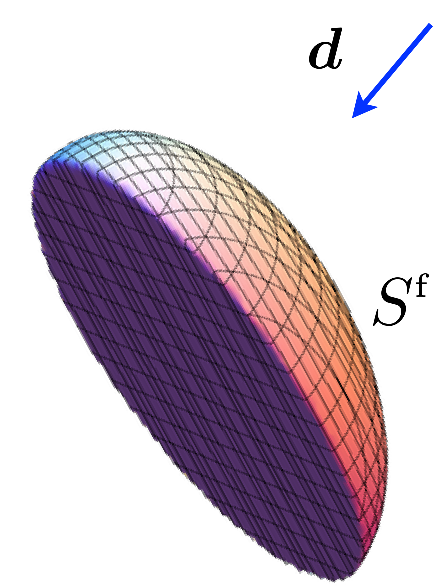


Critical points

$\zeta_0 \in S^{\rm f}$ where $\nabla_{\rm S}\varphi = \mathbf{0}$ (stationary pts)

$\zeta_0 \in S^{\rm f}$ where f or φ fail to be differentiable

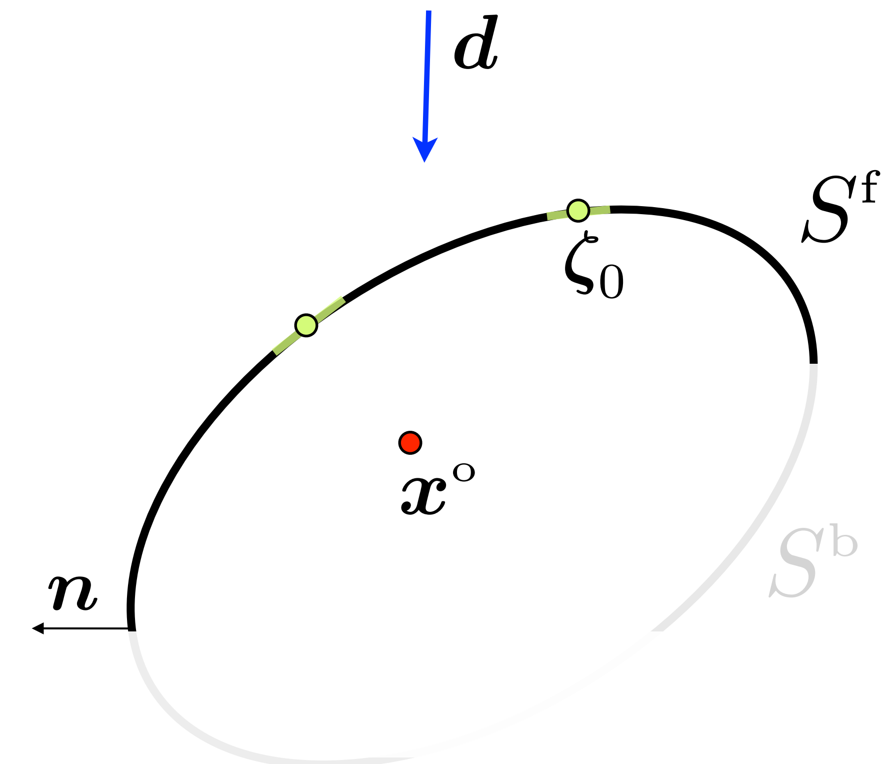
$\forall \zeta_0 \in \partial S^{\rm f}$ $\quad \int_{S^{\rm f}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\zeta)}{8\pi r}\left(1+\frac{\mathrm{i}}{kr}\right)\widehat{(\boldsymbol{x}^{\circ}-\zeta)}\, e^{\mathrm{i}k(\zeta\cdot\boldsymbol{d}+r)}$



Oscillatory integral

"slow"   "fast"

$$\mathsf{T}(\boldsymbol{x}^{\text{o}}) \;=\; \int_{S^{\text{f}}} f(\zeta)\, e^{\mathrm{i}k\varphi(\zeta)}\, \mathrm{d}\zeta$$



Critical points

$\zeta_0 \in S^{\text{f}}$ where $\nabla_{\text{S}}\varphi = \mathbf{0}$ (stationary pts)

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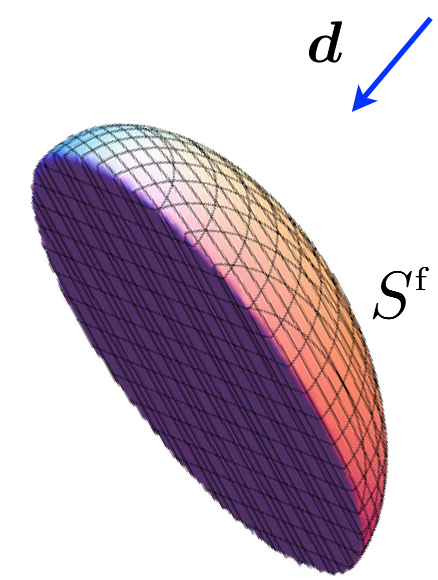


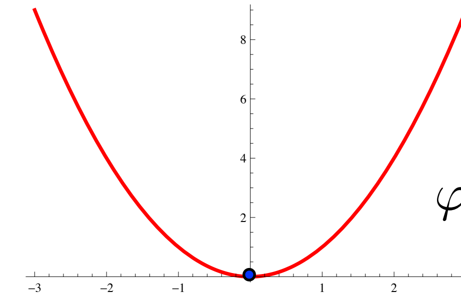
$\forall \zeta_0 \in \partial S^{\text{f}}$ $\quad \int_{S^{\text{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\zeta)}{8\pi r}\left(1+\frac{\mathrm{i}}{kr}\right)\widehat{(\boldsymbol{x}^{\text{o}}-\zeta)}\, e^{\mathrm{i}k(\zeta\cdot\boldsymbol{d}+r)}$

Stationary Phase (MSP)

1D integral

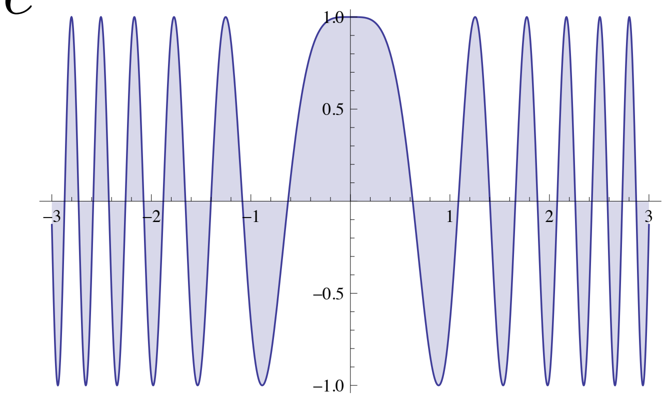
$$\int_{-\infty}^{\infty} f(x)\, e^{\mathrm{i}k\varphi(x)}\, \mathrm{d}x \;=\; e^{\mathrm{i}k\varphi_0} \sum_{n=0}^{\infty} C_n\, k^{-(n+1/2)}$$

$$C_0 = \frac{\sqrt{2\pi}\, f(x_0)}{\sqrt{|\varphi''(x_0)|}}\, e^{\mathrm{i}\pi\delta/4}, \qquad \delta = \mathrm{sign}[\varphi''(x_0)]$$



$\varphi(x) = x^2$

$e^{\mathrm{i}kx^2}$



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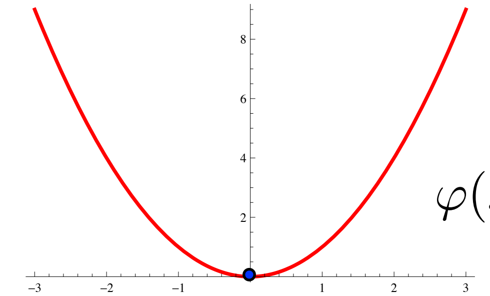

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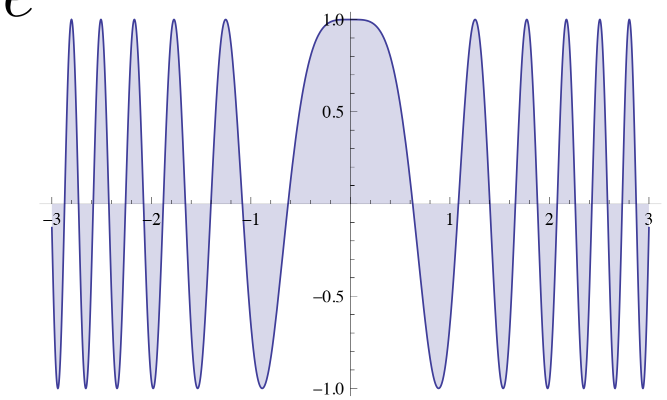
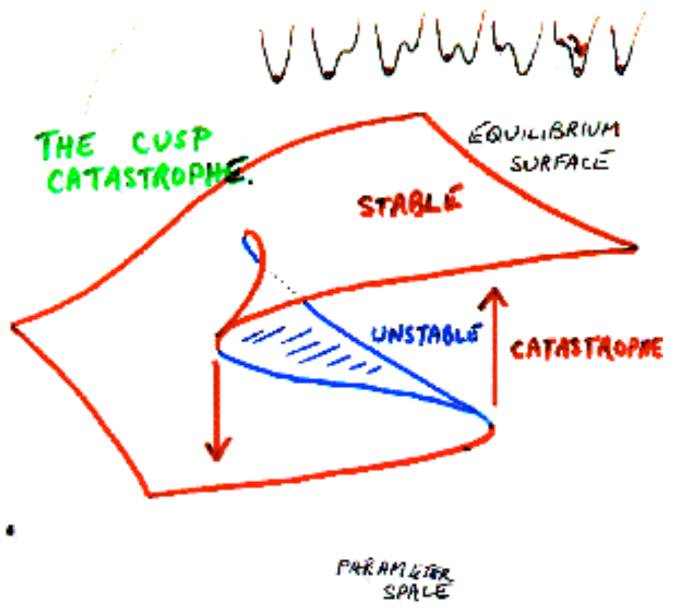


Catastrophe theory

Poincare, Thom (1960's), Poston & Stewart (1978)



Morse lemma

$\varphi : \mathbb{R}^n \to \mathbb{R}$ smooth

$\nabla\varphi|_{\boldsymbol{x}=\boldsymbol{x}_0} \;=\; \mathbf{0}$ critical point

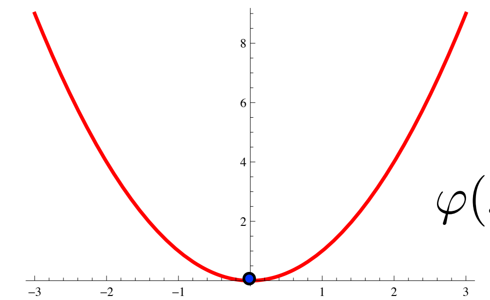
$\det(\boldsymbol{H})|_{\boldsymbol{x}=\boldsymbol{x}_0} \neq 0, \qquad H_{ij} = \dfrac{\partial^2\varphi}{\partial x_i \partial x_j}$ non-degenerate

$$\exists\, \boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}) : \qquad \varphi(\boldsymbol{x}) \;=\; \varphi(\boldsymbol{x}_0) \pm \xi_1^2 \;\ldots\; \pm \xi_n^2 \qquad \forall \boldsymbol{x} \in \mathcal{N}_{\boldsymbol{x}_0}$$

Morse saddle

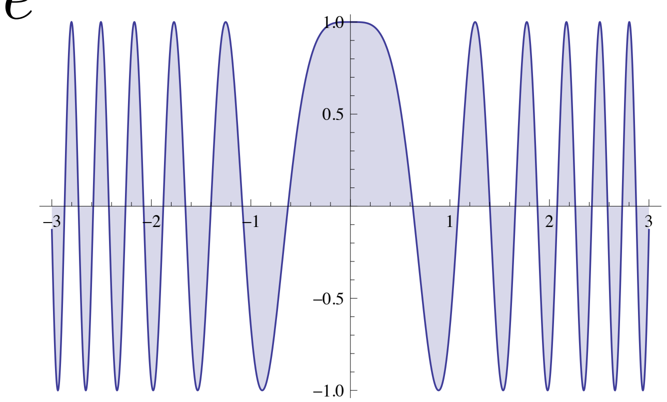
Stationary phase

1D integral



$\varphi(x) = x^2$

e^{ikx^2}

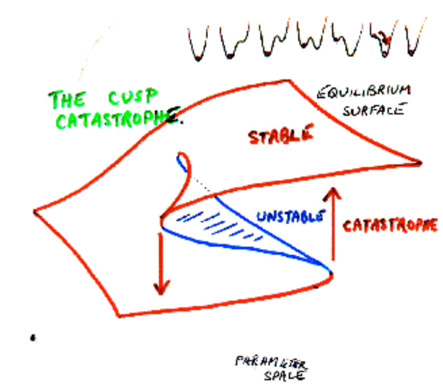


$$\int_{-\infty}^{\infty} f(x)\, e^{ik\varphi(x)}\, \mathrm{d}x \;=\; e^{ik\varphi_0} \sum_{n=0}^{\infty} C_n\, k^{-(n+1/2)}$$

$$C_0 = \frac{\sqrt{2\pi}\, f(x_0)}{\sqrt{|\varphi''(x_0)|}}\, e^{i\pi\delta/4}, \qquad \delta = \mathrm{sign}[\varphi''(x_0)]$$

Catastrophe theory

Poincare, Thom (1960's), Poston & Stewart (1978)



Splitting lemma

$$\varphi : \mathbb{R}^n \to \mathbb{R}$$ smooth

$$\nabla\varphi|_{\boldsymbol{x}=\boldsymbol{x}_0} \;=\; \mathbf{0}$$ critical point

$$\mathrm{rank}(\boldsymbol{H})|_{\boldsymbol{x}=\boldsymbol{x}_0} = p < n, \qquad H_{ij} = \frac{\partial^2\varphi}{\partial x_i \partial x_j}$$ degenerate

Stationary phase

1D integral

$$\int_{-\infty}^{\infty} f(x)\, e^{\mathrm{i}k\varphi(x)}\, \mathrm{d}x \;=\; e^{\mathrm{i}k\varphi_0} \sum_{n=0}^{\infty} C_n\, k^{-(n+1/2)}$$

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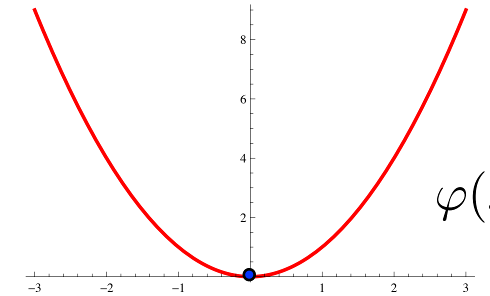

$\varphi(x) = x^2$

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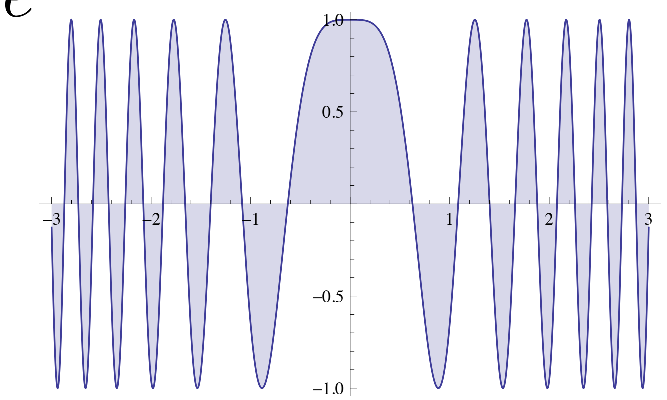

Catastrophe theory

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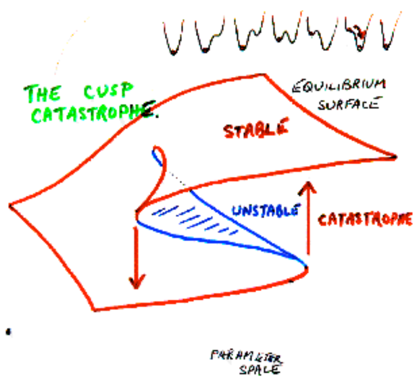


Splitting lemma

$\varphi : \mathbb{R}^n \to \mathbb{R}$ — smooth

$\nabla\varphi|_{\boldsymbol{x}=\boldsymbol{x}_0} = \mathbf{0}$ — critical point

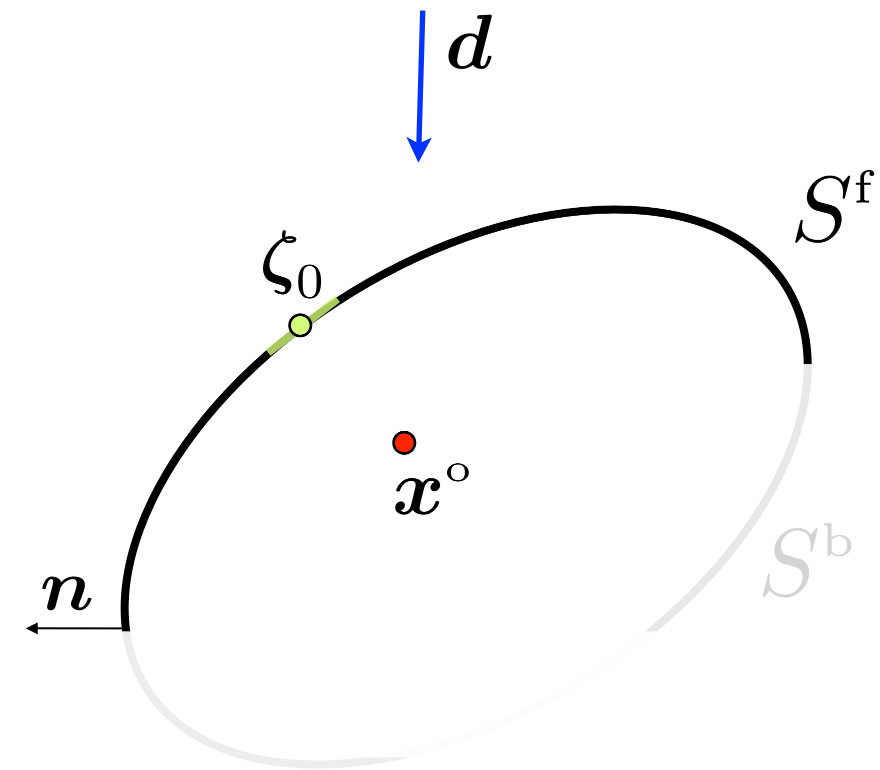
$\mathrm{rank}(\boldsymbol{H})|_{\boldsymbol{x}=\boldsymbol{x}_0} = p < n, \qquad H_{ij} = \dfrac{\partial^2\varphi}{\partial x_i \partial x_j}$ — degenerate

smooth

$$\exists\, \boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}) : \qquad \varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x}_0) \pm \xi_1^2 \ldots \pm \xi_p^2 + \hat{f}(\xi_{p+1}, \ldots, \xi_n) \qquad \forall \boldsymbol{x} \in \mathcal{N}_{\boldsymbol{x}_0}$$

Surface integral

$$\mathsf{T}(\boldsymbol{x}^{\circ}) \;=\; \int_{S^{\mathrm{f}}} f(\boldsymbol{\zeta})\, e^{\mathrm{i}k\varphi(\boldsymbol{\zeta})}\,\mathrm{d}\boldsymbol{\zeta}$$

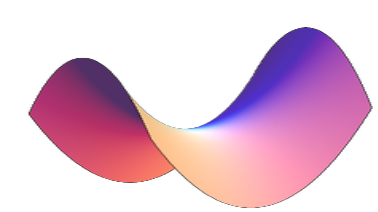


non-degenerate stationary point

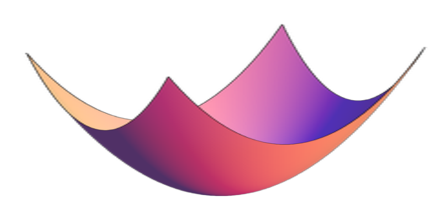
$$\int_{S^{\mathrm{f}}} f(\boldsymbol{\zeta})\, e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r)}\,\mathrm{d}S_{\eta}, \qquad \boldsymbol{\zeta} = \boldsymbol{\zeta}(\eta^1, \eta^2), \qquad r = |\boldsymbol{\zeta} - \boldsymbol{x}^{\circ}|$$

e.g. Blestein & Handelsman (1986)

$$\frac{2\pi}{k}\,\frac{f(\boldsymbol{\zeta}_0)}{\sqrt{|\det \boldsymbol{H}|}}\, e^{\mathrm{i}k(\boldsymbol{\zeta}_0\cdot\boldsymbol{d} \pm r_0) + \mathrm{i}(\mathrm{sgn}\,\boldsymbol{H})\pi/4}$$

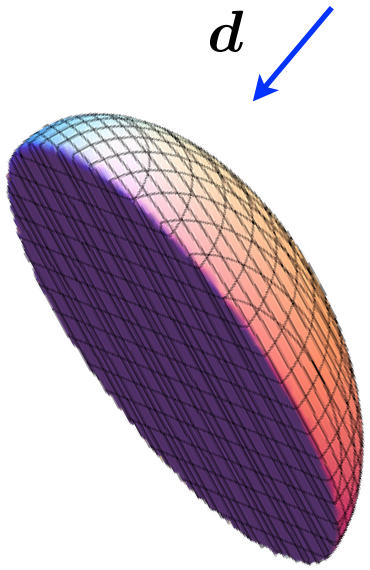


$$H_{ij} \;=\; \left.\frac{\partial^2(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r)}{\partial\eta^i\partial\eta^j}\right|_{\boldsymbol{\zeta}=\boldsymbol{\zeta}_0}$$



Stationary points

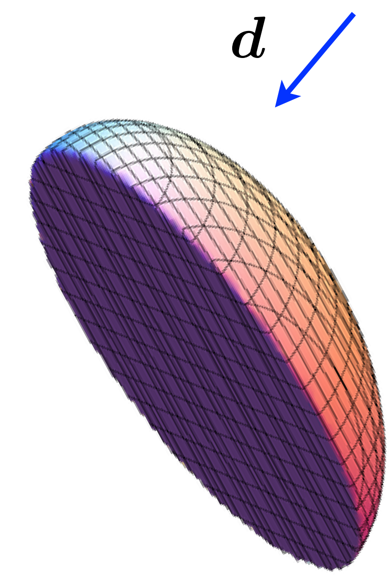
$$\nabla_\eta(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r) = \mathbf{0} \qquad \Longrightarrow \qquad \left[\boldsymbol{d} \pm (\widehat{\boldsymbol{\zeta} - \boldsymbol{x}^\circ})\right]\cdot\frac{\partial\boldsymbol{\zeta}}{\partial\eta^p} = 0, \qquad p = 1, 2$$



$$\boldsymbol{\zeta}_{\rm I}^{\pm} = \boldsymbol{x}^\circ \mp r\,\boldsymbol{d},$$
$$\boldsymbol{\zeta}_{\rm II}^{\pm} = \boldsymbol{x}^\circ \mp r\left[\boldsymbol{d} + 2|\boldsymbol{d}\cdot\boldsymbol{n}|\,\boldsymbol{n}(\boldsymbol{\zeta}_{\rm II}^{\pm})\right]$$

Stationary points

$$\nabla_\eta(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r) = \boldsymbol{0} \qquad \Longrightarrow \qquad \left[\boldsymbol{d} \pm \widehat{(\boldsymbol{\zeta} - \boldsymbol{x}^\circ)}\right]\cdot \frac{\partial \boldsymbol{\zeta}}{\partial \eta^p} \;=\; 0, \qquad p = 1, 2$$



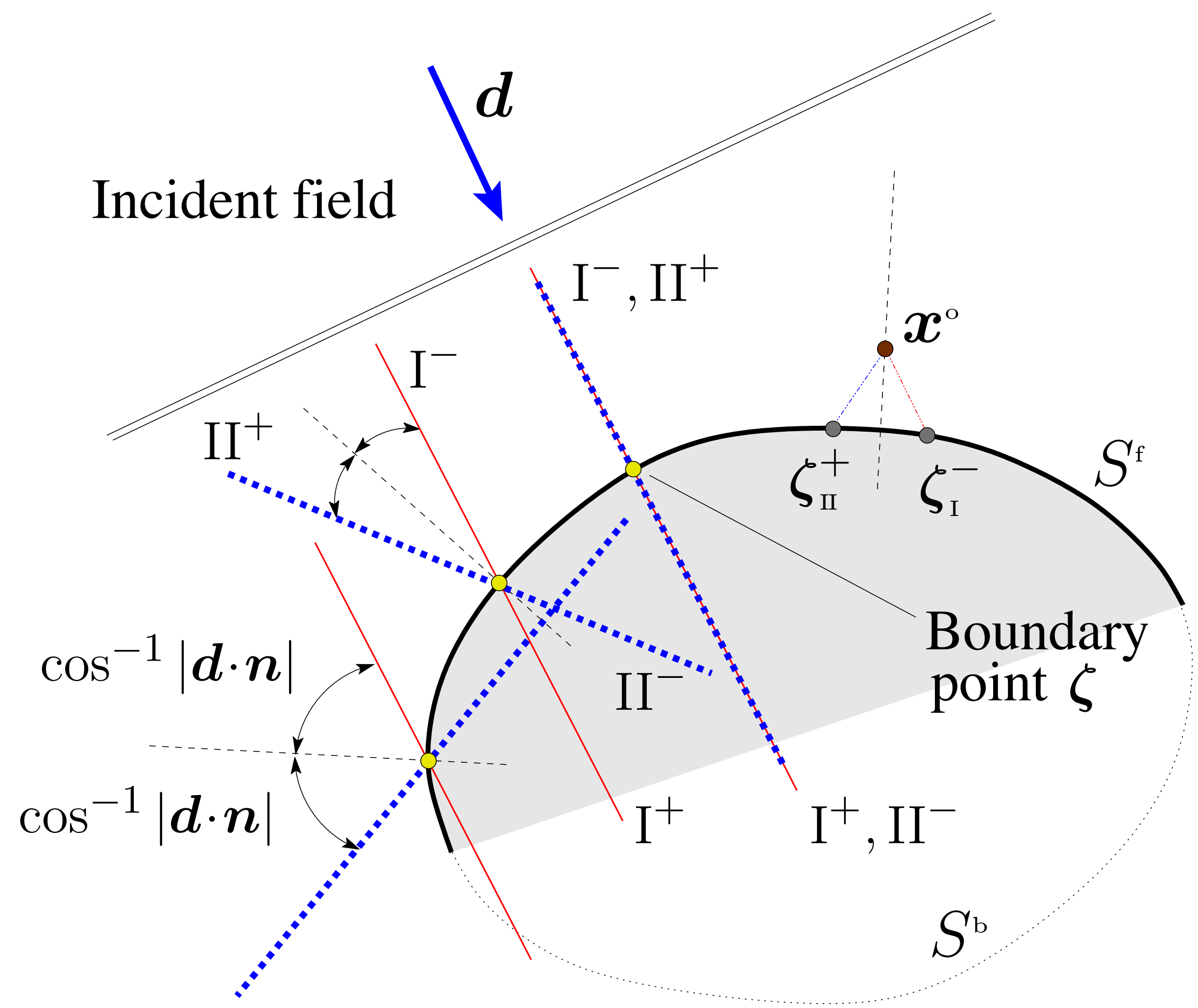
$$\boldsymbol{\zeta}_{\rm I}^{\pm} = \boldsymbol{x}^\circ \mp r\,\boldsymbol{d},$$

$$\boldsymbol{\zeta}_{\rm II}^{\pm} = \boldsymbol{x}^\circ \mp r\left[\boldsymbol{d} + 2|\boldsymbol{d}\cdot\boldsymbol{n}|\,\boldsymbol{n}(\boldsymbol{\zeta}_{\rm II}^{\pm})\right]$$

or

$$\mathrm{I}^{\pm} \;:=\; \{\boldsymbol{x}^\circ:\; \boldsymbol{x}^\circ = \boldsymbol{\zeta}_{\rm I}^{\pm} \pm r\,\boldsymbol{d}, \quad r > 0\},$$

$$\mathrm{II}^{\pm} \;:=\; \{\boldsymbol{x}^\circ:\; \boldsymbol{x}^\circ = \boldsymbol{\zeta}_{\rm II}^{\pm} \pm r\left[\boldsymbol{d} + 2|\boldsymbol{d}\cdot\boldsymbol{n}|\,\boldsymbol{n}(\boldsymbol{\zeta}_{\rm II}^{\pm})\right], \quad r > 0\}$$



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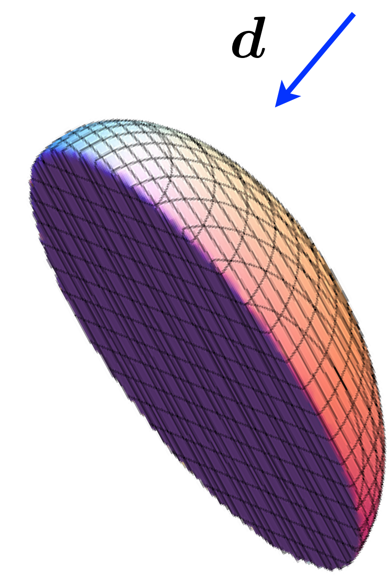

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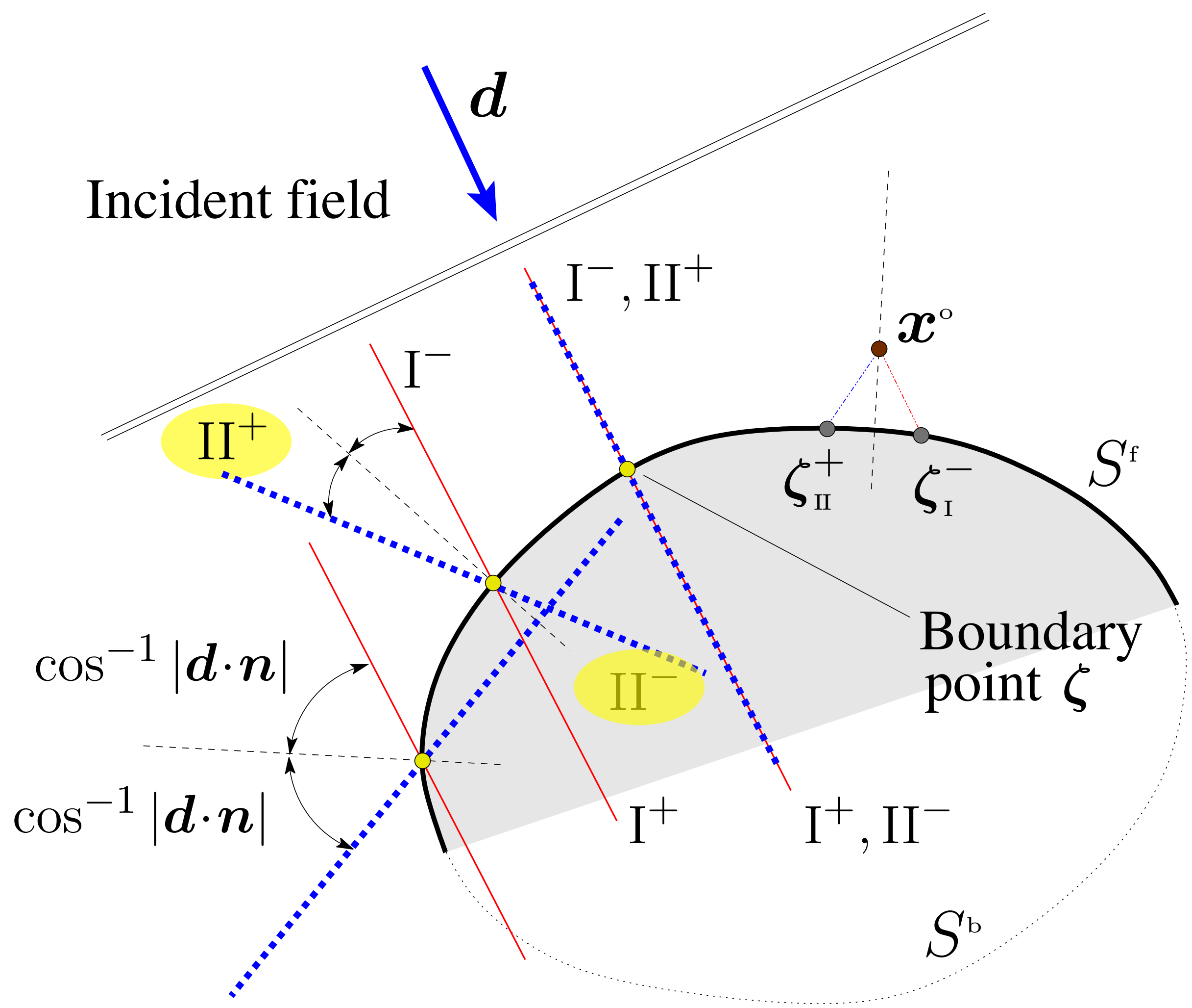
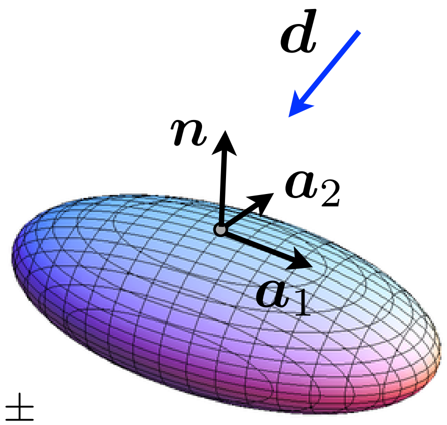


Hessian

$$H_{pq}(\boldsymbol{\zeta}) = \left[\frac{\partial^2(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r)}{\partial\eta^p\partial\eta^q}\right], \qquad r = |\boldsymbol{\zeta} - \boldsymbol{x}^\circ|$$

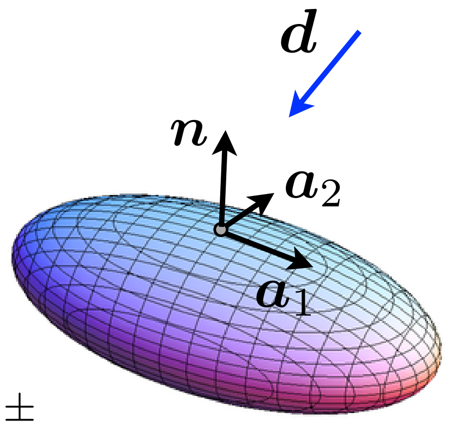


Hessian (principal directions)

$$\boldsymbol{H} = \begin{bmatrix} \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2] + \frac{2}{\rho_1}|\boldsymbol{d}\cdot\boldsymbol{n}| & \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) \\ \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) & \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2)] + \frac{2}{\rho_2}|\boldsymbol{d}\cdot\boldsymbol{n}| \end{bmatrix}, \qquad \begin{matrix} \zeta = \zeta_{\mathrm{I}}^{\pm} \\ \zeta = \zeta_{\mathrm{II}}^{\pm} \end{matrix}$$

Hessian

$$H_{pq}(\boldsymbol{\zeta}) = \left[\frac{\partial^2(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r)}{\partial\eta^p\partial\eta^q}\right], \qquad r = |\boldsymbol{\zeta}-\boldsymbol{x}^\circ|$$



Hessian (principal directions)

$$\boldsymbol{H} = \begin{bmatrix} \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2] + \frac{2}{\rho_1}|\boldsymbol{d}\cdot\boldsymbol{n}| & \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) \\ \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) & \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2)] + \frac{2}{\rho_2}|\boldsymbol{d}\cdot\boldsymbol{n}| \end{bmatrix},$$

$\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathrm{I}}^{\pm}$

$\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathrm{II}}^{\pm}$

Determinant

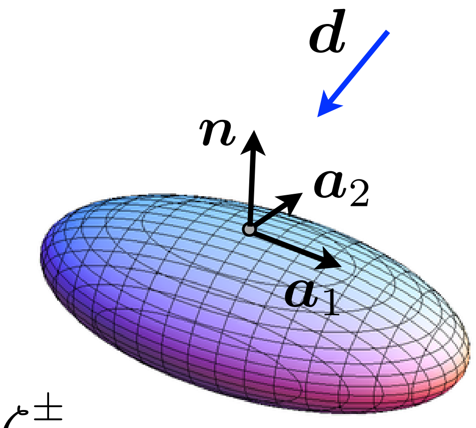
$$\det(\boldsymbol{H}) = \frac{(\boldsymbol{d}\cdot\boldsymbol{n})^2}{r^2} > 0, \qquad \operatorname{sgn}(\boldsymbol{H}) = \pm 2$$

Ray I+

Ray I−

Hessian

$$H_{pq}(\boldsymbol{\zeta}) = \left[\frac{\partial^2(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r)}{\partial\eta^p\partial\eta^q}\right], \qquad r = |\boldsymbol{\zeta} - \boldsymbol{x}^\circ|$$



Hessian (principal directions)

$$\boldsymbol{H} = \begin{bmatrix} \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2] + \frac{2}{\rho_1}|\boldsymbol{d}\cdot\boldsymbol{n}| & \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) \\ \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) & \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2)] + \frac{2}{\rho_2}|\boldsymbol{d}\cdot\boldsymbol{n}| \end{bmatrix}, \qquad \begin{matrix} \boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathrm{I}}^{\pm} \\ \boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathrm{II}}^{\pm} \end{matrix}$$

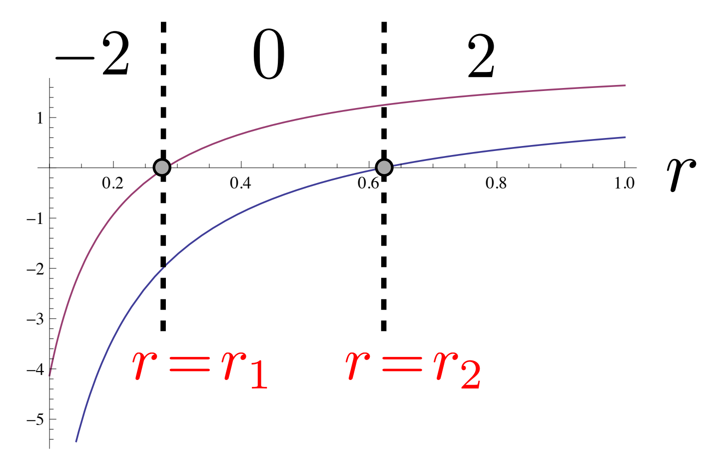
Determinant

$$\det(\boldsymbol{H}) = \frac{(\boldsymbol{d}\cdot\boldsymbol{n})^2}{r^2} > 0, \qquad \operatorname{sgn}(\boldsymbol{H}) = \pm 2$$

Ray I$^+$

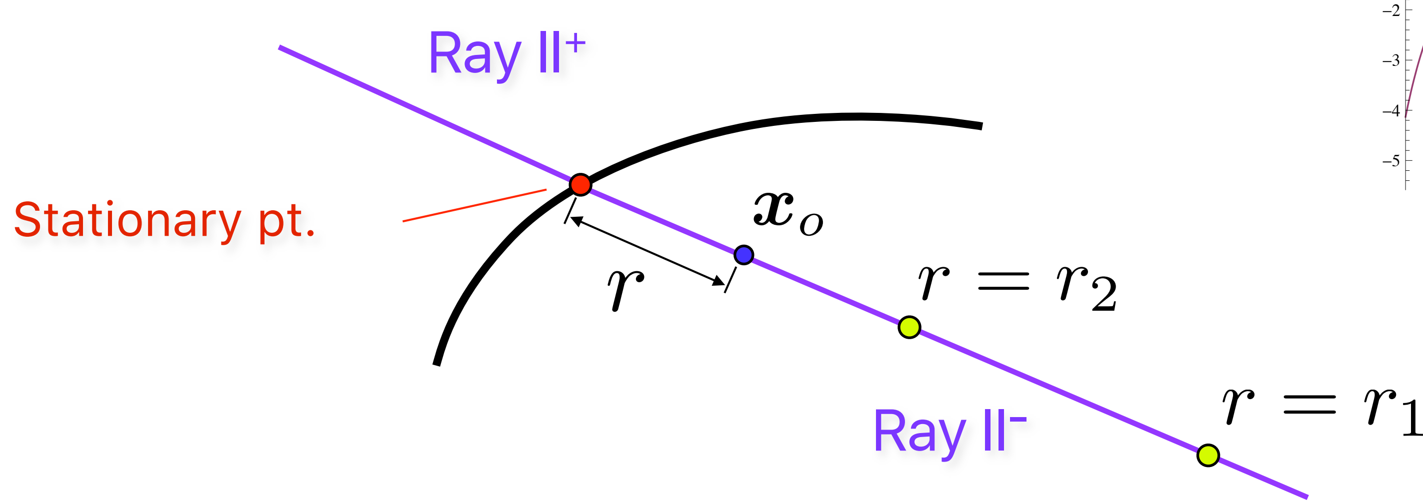
Ray I$^-$

$$\det(\boldsymbol{H}) = \frac{4(\boldsymbol{d}\cdot\boldsymbol{n})^2}{\rho_1\rho_2\, r^2}(r \pm r_1)(r \pm r_2),$$



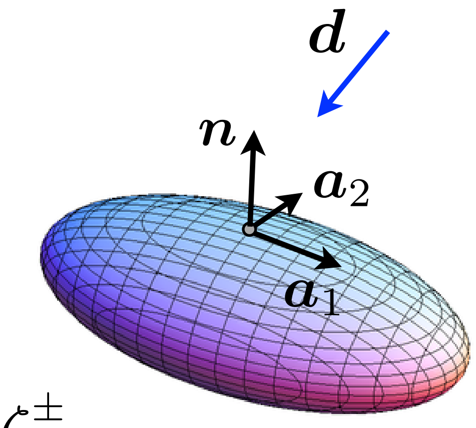
Ray II$^+$

Ray II$^-$



Hessian

$$H_{pq}(\zeta) = \left[\frac{\partial^2(\boldsymbol{\zeta}\cdot\boldsymbol{d} \pm r)}{\partial\eta^p\partial\eta^q}\right], \qquad r = |\boldsymbol{\zeta} - \boldsymbol{x}^{\circ}|$$



Hessian (principal directions)

$$\boldsymbol{H} = \begin{bmatrix} \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2] + \frac{2}{\rho_1}|\boldsymbol{d}\cdot\boldsymbol{n}| & \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) \\ \mp\frac{1}{r}(\boldsymbol{d}\cdot\boldsymbol{a}_1)(\boldsymbol{d}\cdot\boldsymbol{a}_2) & \pm\frac{1}{r}[1-(\boldsymbol{d}\cdot\boldsymbol{a}_1)^2)] + \frac{2}{\rho_2}|\boldsymbol{d}\cdot\boldsymbol{n}| \end{bmatrix}, \qquad \begin{matrix} \zeta = \zeta_{\mathrm{I}}^{\pm} \\ \zeta = \zeta_{\mathrm{II}}^{\pm} \end{matrix}$$

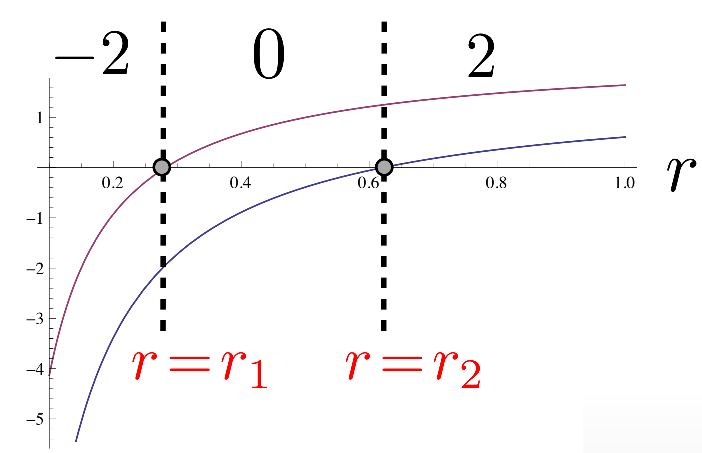
Determinant

$$\det(\boldsymbol{H}) = \frac{(\boldsymbol{d}\cdot\boldsymbol{n})^2}{r^2} > 0, \qquad \mathrm{sgn}(\boldsymbol{H}) = \pm 2$$

Ray I⁺

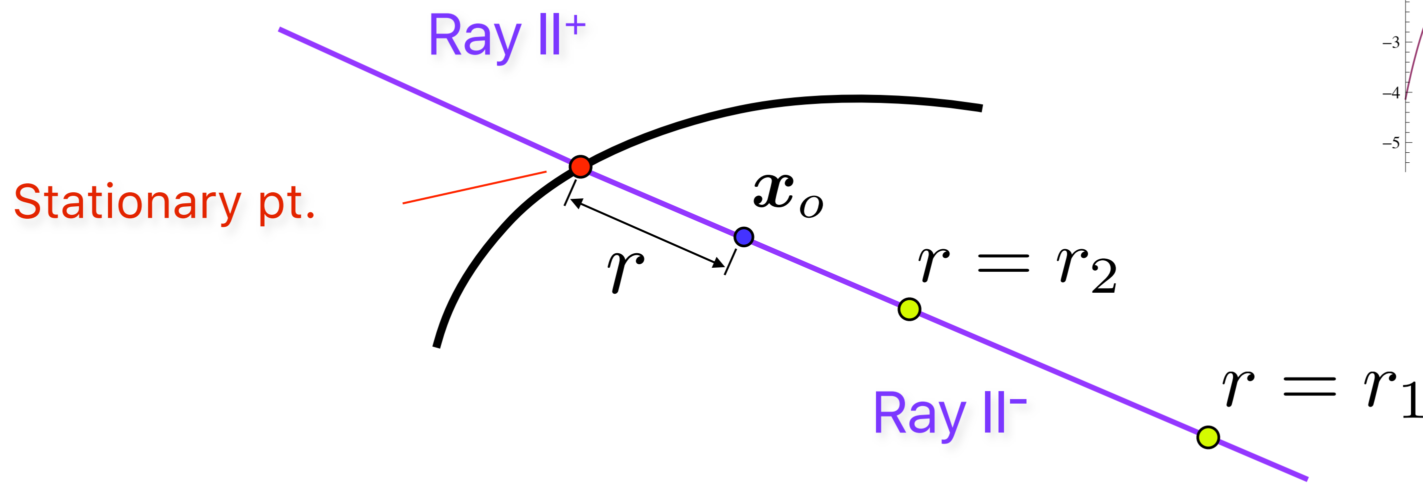
Ray I⁻

$$\det(\boldsymbol{H}) = \frac{4(\boldsymbol{d}\cdot\boldsymbol{n})^2}{\rho_1\rho_2\, r^2}(r \pm r_1)(r \pm r_2),$$



Ray II⁺

Ray II⁻

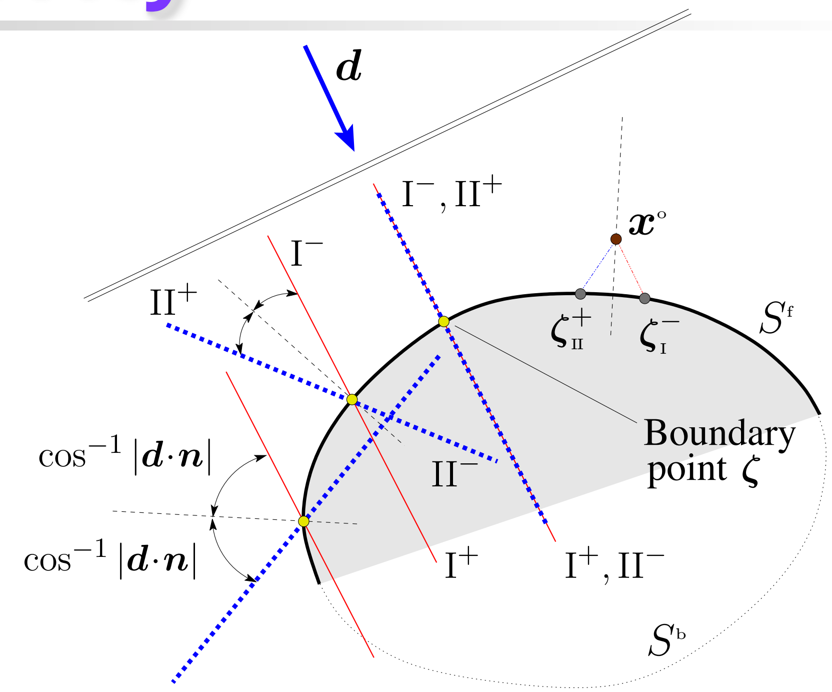


$$\mathrm{rank}(\boldsymbol{H}) \geqslant 1$$

except when

$$|\boldsymbol{d}\cdot\boldsymbol{n}|^2 = \rho_2/\rho_1 = 1.$$

Topological Sensitivity



Non-degenerate stationary pt

$$\mathsf{T}^{\mathrm{I}^\pm} = \mp \frac{3(1-\beta)}{2(2+\beta)} \frac{1}{r_{\mathrm{I}}^\pm}, \qquad \boldsymbol{x}^\circ \in \mathrm{I}^\pm, \qquad r_{\mathrm{I}}^\pm = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}_{\mathrm{I}}^\pm|$$

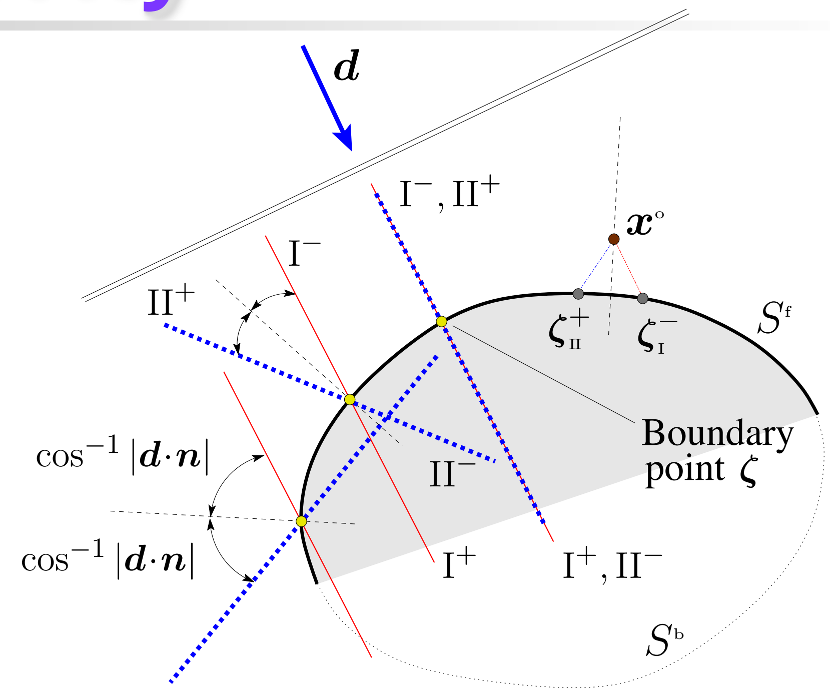
"outside"

$$\mathsf{T}^{\mathrm{II}^+} = -\frac{3(1-\beta)}{4(2+\beta)} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{(r_{\mathrm{II}}^+ + r_1)(r_{\mathrm{II}}^+ + r_2)}} \big(1-2(\boldsymbol{d}\cdot\boldsymbol{n})^2\big)\, \mathrm{Im}\left\{ \left(k + \frac{\mathrm{i}}{r_{\mathrm{II}}^+}\right) e^{2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^+} \right\}$$
$$- \frac{1-\beta\gamma^2}{4} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{(r_{\mathrm{II}}^+ + r_1)(r_{\mathrm{II}}^+ + r_2)}}\, k\, \mathrm{Im}\left\{ e^{2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^+} \right\}, \qquad \boldsymbol{x}^\circ \in \mathrm{II}^+, \qquad r_{\mathrm{II}}^+ = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}_{\mathrm{II}}^+|$$

"inside"

$$\mathsf{T}^{\mathrm{II}^-} = \frac{3(1-\beta)}{4(2+\beta)} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{|(r_{\mathrm{II}}^- - r_1)(r_{\mathrm{II}}^- - r_2)|}} \big(1-2(\boldsymbol{d}\cdot\boldsymbol{n})^2\big)\, \mathrm{Im}\left\{ \left(k - \frac{\mathrm{i}}{r_{\mathrm{II}}^-}\right) e^{-2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^- + \mathrm{i}(\mathrm{sgn}\,\boldsymbol{H} - 2)\pi/4} \right\}$$
$$+ \frac{1-\beta\gamma^2}{4} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{|(r_{\mathrm{II}}^- - r_1)(r_{\mathrm{II}}^- - r_2)|}}\, k\, \mathrm{Im}\left\{ e^{-2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^- + \mathrm{i}(\mathrm{sgn}\, H_{ij} - 2)\pi/4} \right\}, \qquad \boldsymbol{x}^\circ \in \mathrm{II}^-, \qquad r_{\mathrm{II}}^- = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}_{\mathrm{II}}^-|$$

Topological Sensitivity



Non-degenerate stationary pt

$$\mathsf{T}^{\mathrm{I}^\pm} = \mp \frac{3(1-\beta)}{2(2+\beta)} \frac{1}{r_{\mathrm{I}}^\pm}, \qquad \boldsymbol{x}^\circ \in \mathrm{I}^\pm, \qquad r_{\mathrm{I}}^\pm = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}_{\mathrm{I}}^\pm|$$

h.o.t.

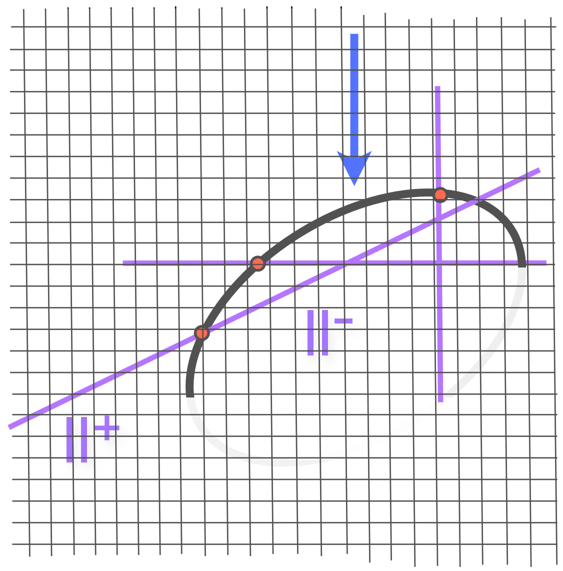
"outside"

$$\mathsf{T}^{\mathrm{II}^+} = -\frac{3(1-\beta)}{4(2+\beta)} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{(r_{\mathrm{II}}^+ + r_1)(r_{\mathrm{II}}^+ + r_2)}} \big(1-2(\boldsymbol{d}\cdot\boldsymbol{n})^2\big)\, \mathrm{Im}\left\{\left(k + \frac{\mathrm{i}}{r_{\mathrm{II}}^+}\right) e^{2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^+}\right\}$$
$$- \frac{1-\beta\gamma^2}{4} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{(r_{\mathrm{II}}^+ + r_1)(r_{\mathrm{II}}^+ + r_2)}}\, k\, \mathrm{Im}\left\{e^{2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^+}\right\}, \qquad \boldsymbol{x}^\circ \in \mathrm{II}^+, \qquad r_{\mathrm{II}}^+ = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}_{\mathrm{II}}^+|$$

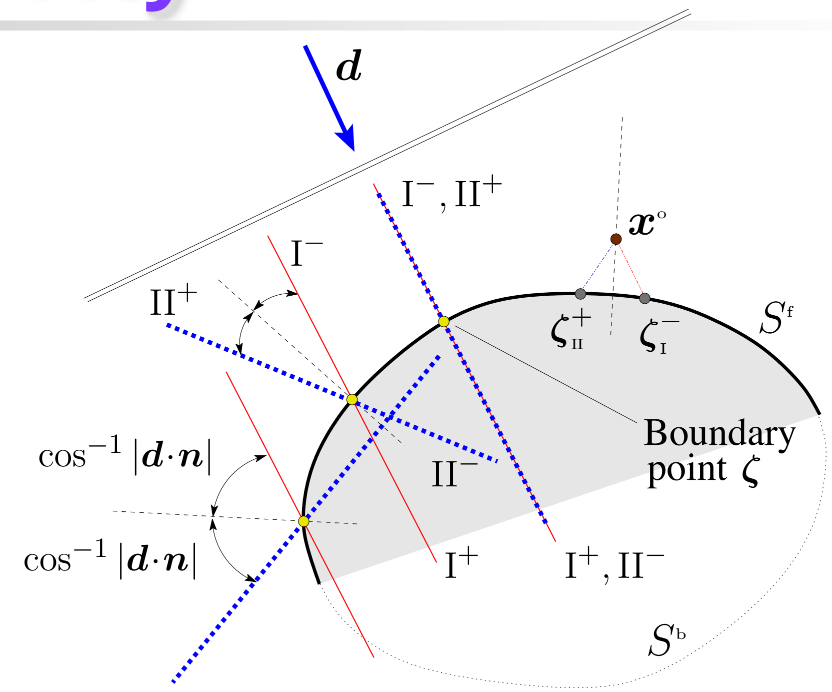
"inside"

$$\mathsf{T}^{\mathrm{II}^-} = \frac{3(1-\beta)}{4(2+\beta)} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{|(r_{\mathrm{II}}^- - r_1)(r_{\mathrm{II}}^- - r_2)|}} \big(1-2(\boldsymbol{d}\cdot\boldsymbol{n})^2\big)\, \mathrm{Im}\left\{\left(k - \frac{\mathrm{i}}{r_{\mathrm{II}}^-}\right) e^{-2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^- + \mathrm{i}(\mathrm{sgn}\,\boldsymbol{H}-2)\pi/4}\right\}$$
$$+ \frac{1-\beta\gamma^2}{4} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{|(r_{\mathrm{II}}^- - r_1)(r_{\mathrm{II}}^- - r_2)|}}\, k\, \mathrm{Im}\left\{e^{-2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^- + \mathrm{i}(\mathrm{sgn}\,H_{ij}-2)\pi/4}\right\}, \qquad \boldsymbol{x}^\circ \in \mathrm{II}^-, \qquad r_{\mathrm{II}}^- = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}_{\mathrm{II}}^-|$$

Topological Sensitivity



TS image



"outside"

$$\mathsf{T}^{\mathrm{II}^+} = -\frac{3(1-\beta)}{4(2+\beta)}\frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{(r_{\mathrm{II}}^+ + r_1)(r_{\mathrm{II}}^+ + r_2)}}\,(1-2(\boldsymbol{d}\cdot\boldsymbol{n})^2)\,\mathrm{Im}\left\{\left(k+\frac{\mathrm{i}}{r_{\mathrm{II}}^+}\right)e^{2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^+}\right\}$$
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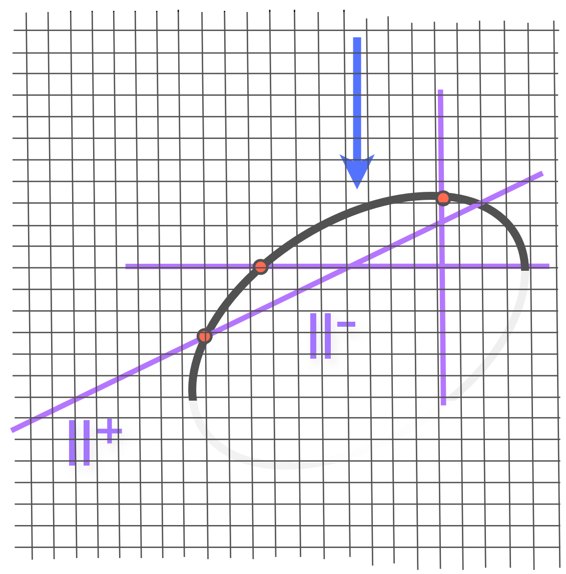
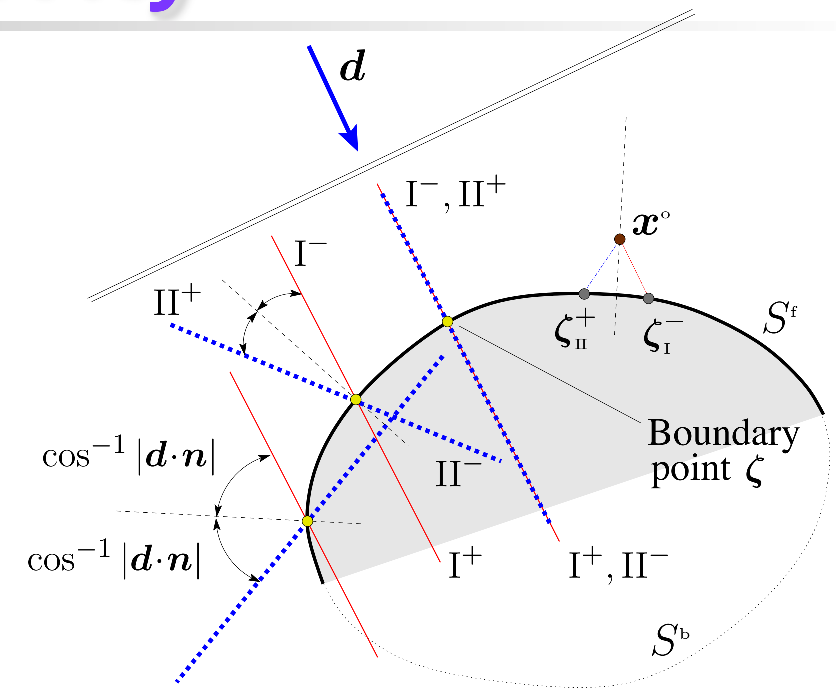
"inside"

$$\mathsf{T}^{\mathrm{II}^-} = \frac{3(1-\beta)}{4(2+\beta)}\frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{|(r_{\mathrm{II}}^- - r_1)(r_{\mathrm{II}}^- - r_2)|}}\,(1-2(\boldsymbol{d}\cdot\boldsymbol{n})^2)\,\mathrm{Im}\left\{\left(k-\frac{\mathrm{i}}{r_{\mathrm{II}}^-}\right)e^{-2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^- + \mathrm{i}(\mathrm{sgn}\,\boldsymbol{H}-2)\pi/4}\right\}$$
$$+\frac{1-\beta\gamma^2}{4}\frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{|(r_{\mathrm{II}}^- - r_1)(r_{\mathrm{II}}^- - r_2)|}}\,k\,\mathrm{Im}\left\{e^{-2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^- + \mathrm{i}(\mathrm{sgn}\,H_{ij}-2)\pi/4}\right\},\qquad \boldsymbol{x}^\circ\in\mathrm{II}^-,\qquad r_{\mathrm{II}}^- = |\boldsymbol{x}^\circ-\boldsymbol{\zeta}_{\mathrm{II}}^-|$$

Topological Sensitivity



TS image



"outside"

$$\mathsf{T}^{\mathrm{II}^+} = -\frac{3(1-\beta)}{4(2+\beta)} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{(r_{\mathrm{II}}^+ + r_1)(r_{\mathrm{II}}^+ + r_2)}} (1-2(\boldsymbol{d}\cdot\boldsymbol{n})^2)\,\mathrm{Im}\left\{\left(k + \frac{\mathrm{i}}{r_{\mathrm{II}}^+}\right) e^{2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^+}\right\}$$

$$-\frac{1-\beta\gamma^2}{4} \frac{\sqrt{\rho_1\,\rho_2}}{\sqrt{(r_{\mathrm{II}}^+ + r_1)(r_{\mathrm{II}}^+ + r_2)}}\, k\,\mathrm{Im}\left\{e^{2\mathrm{i}k(\boldsymbol{d}\cdot\boldsymbol{n})^2 r_{\mathrm{II}}^+}\right\}, \qquad \boldsymbol{x}^\circ \in \mathrm{II}^+, \qquad r_{\mathrm{II}}^+ = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}_{\mathrm{II}}^+|$$

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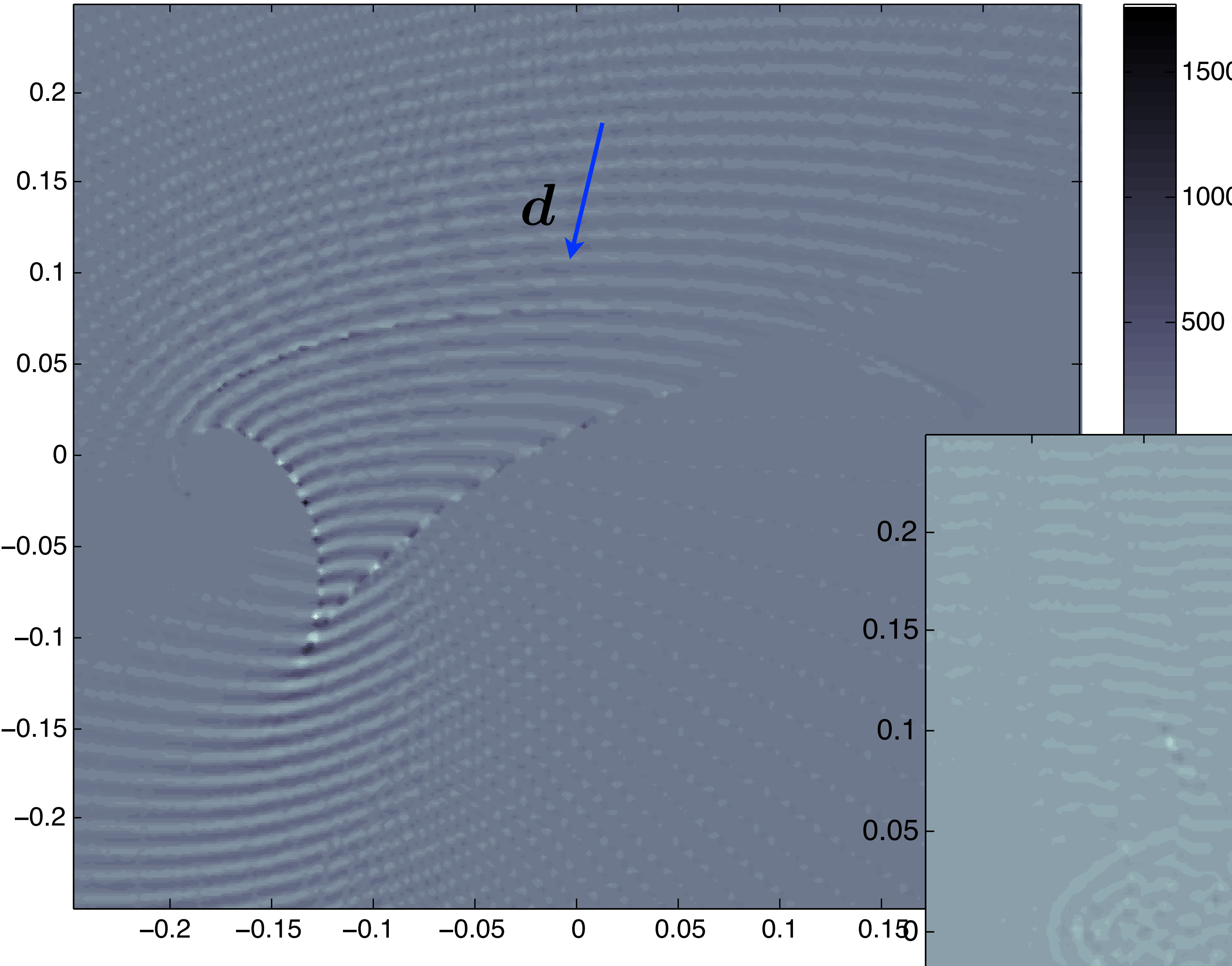
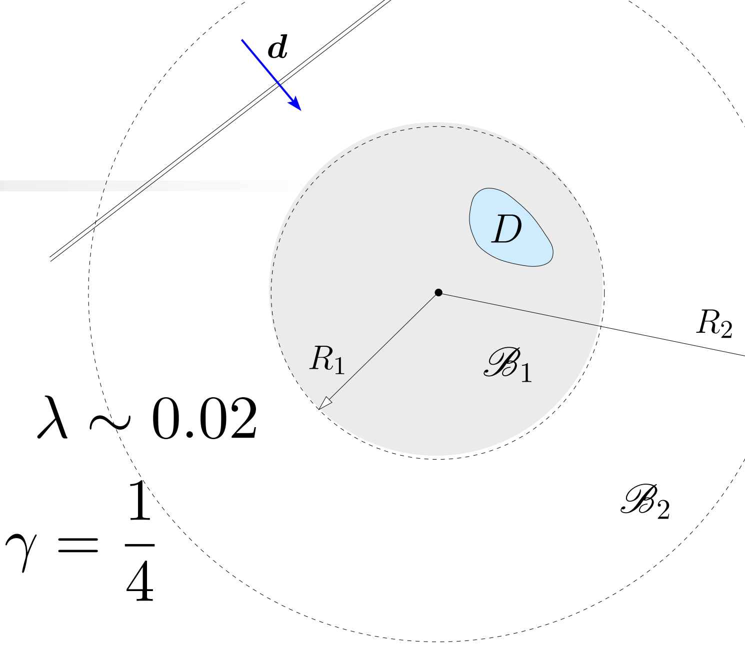
$$\mathsf{T}^{\mathrm{I}^\pm}(\boldsymbol{x}^\circ) = O(1), \quad \mathsf{T}^{\mathrm{II}^\pm}(\boldsymbol{x}^\circ) = O(k), \qquad k\,\ell \gg 1$$

$$\mathsf{T}^{\mathrm{I}^\pm}, \mathsf{T}^{\mathrm{II}^\pm} = O(1/r) \quad \text{as} \quad r \to \infty$$

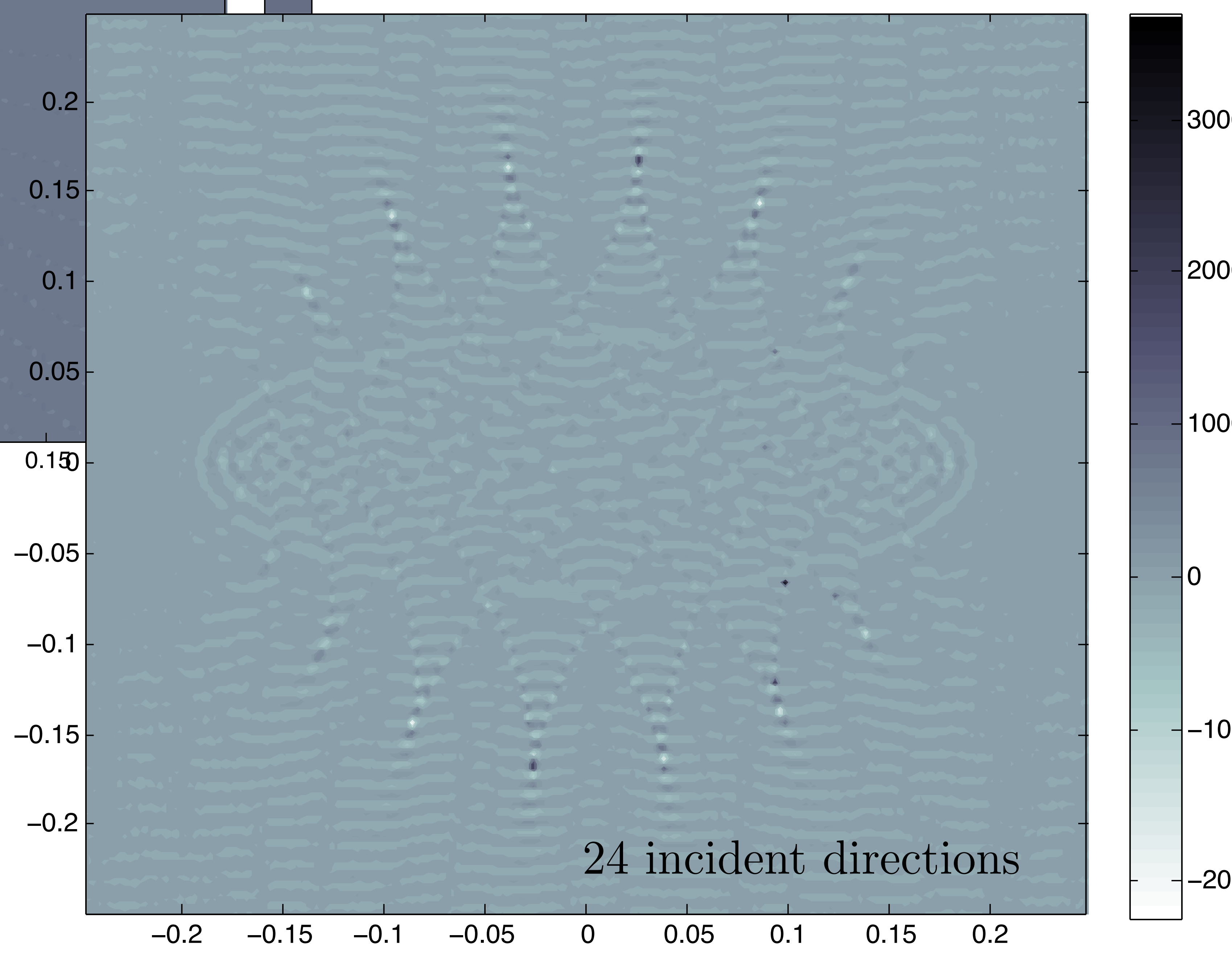
Ignorance is a bliss





$$k = 300, \quad \lambda \sim 0.02$$

$$\beta = 4, \quad \gamma = \frac{1}{4}$$



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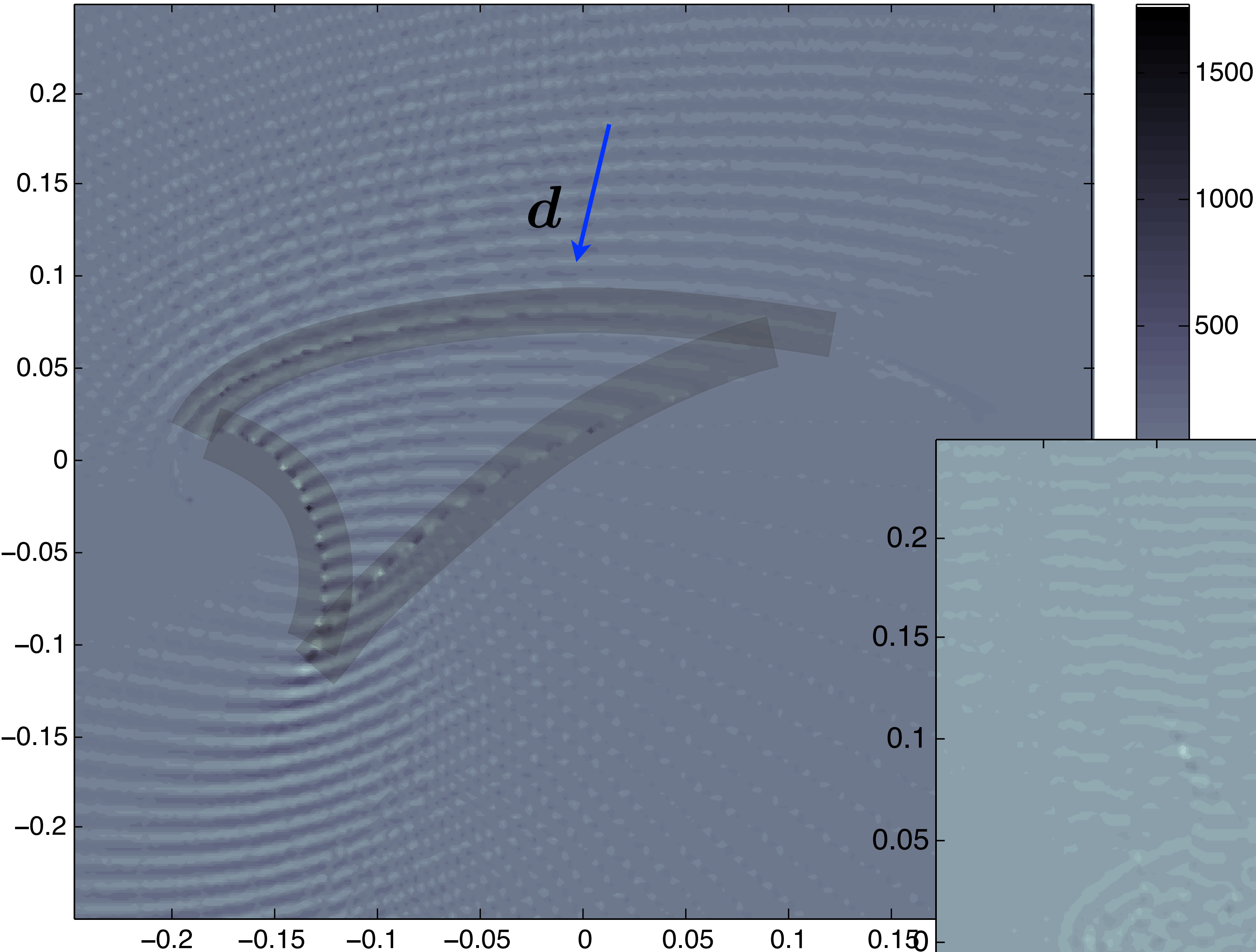

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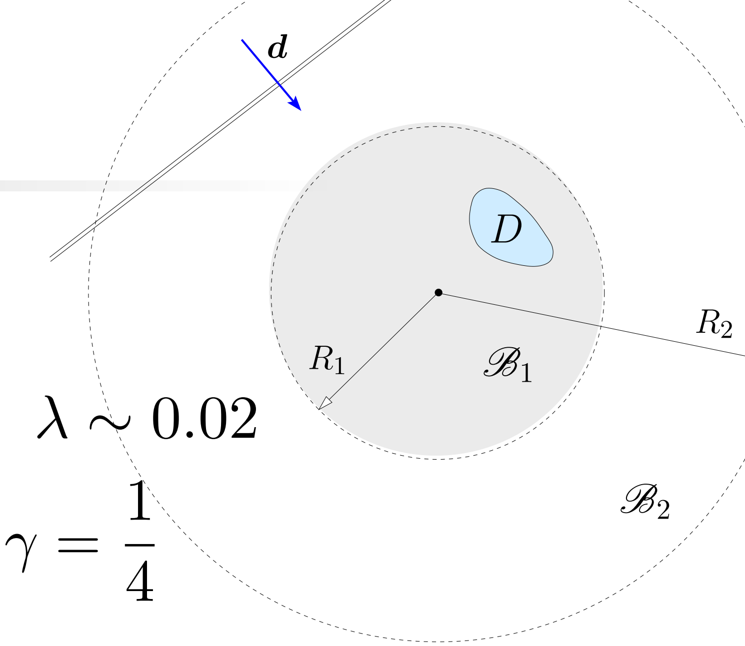
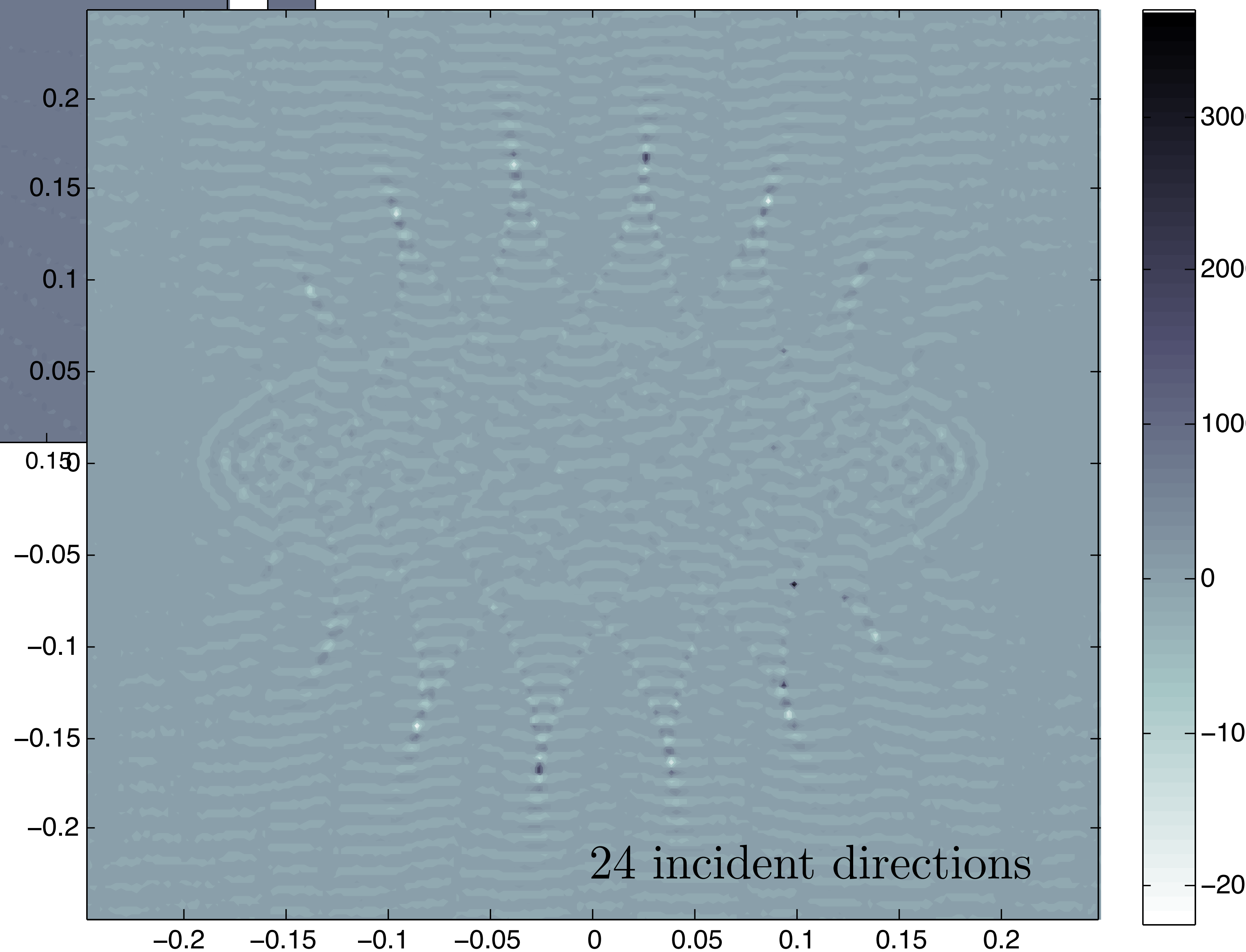
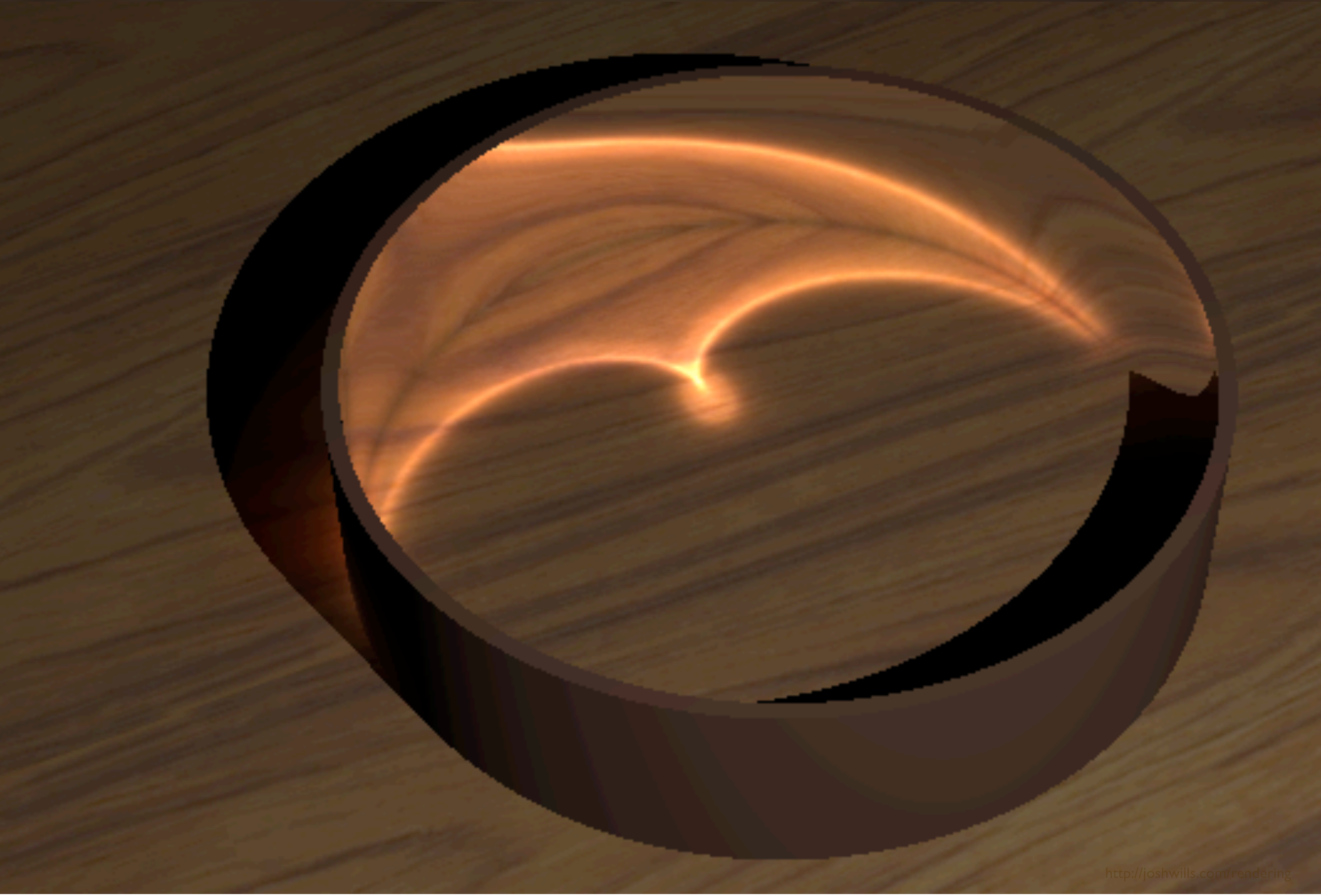


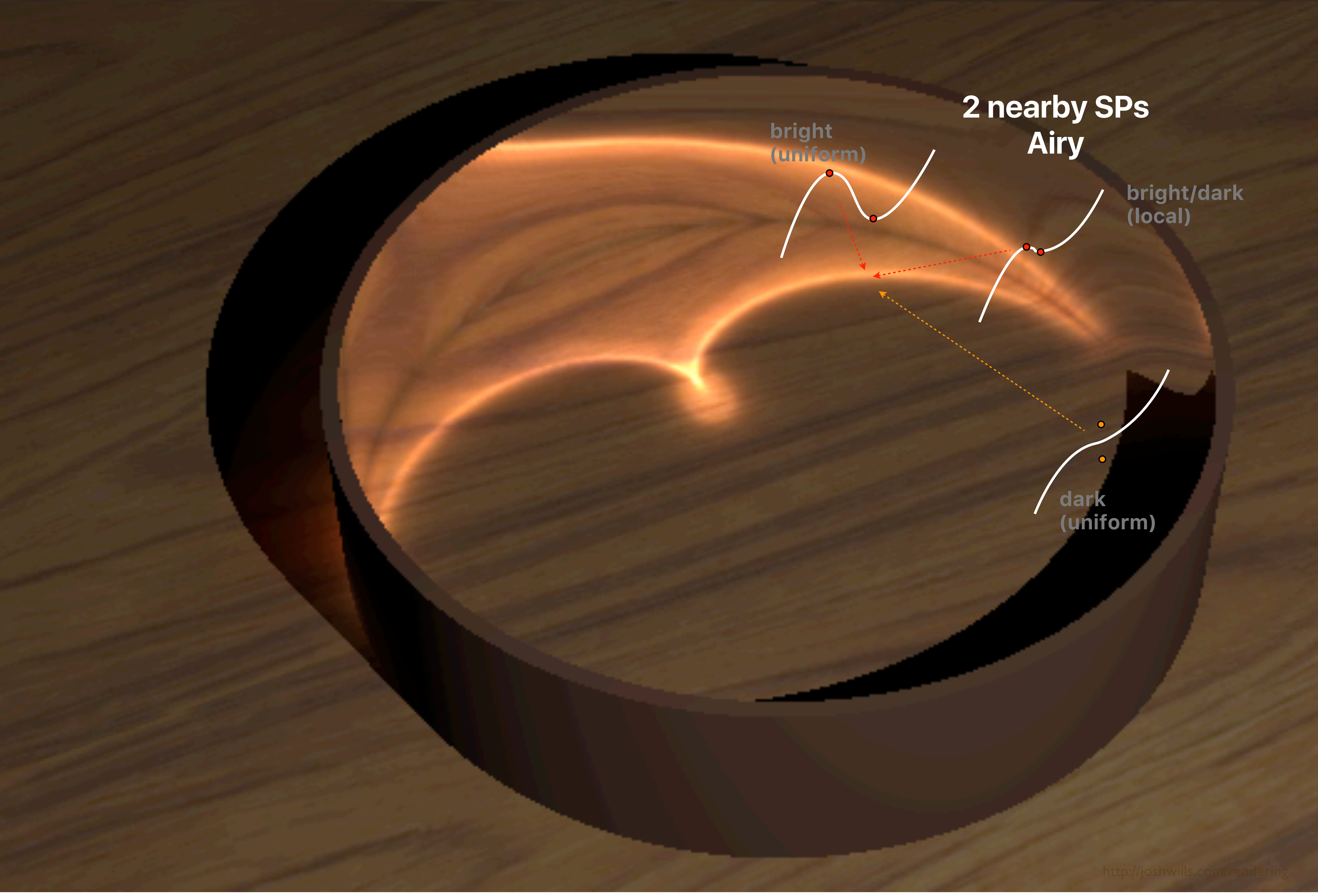


Caustics



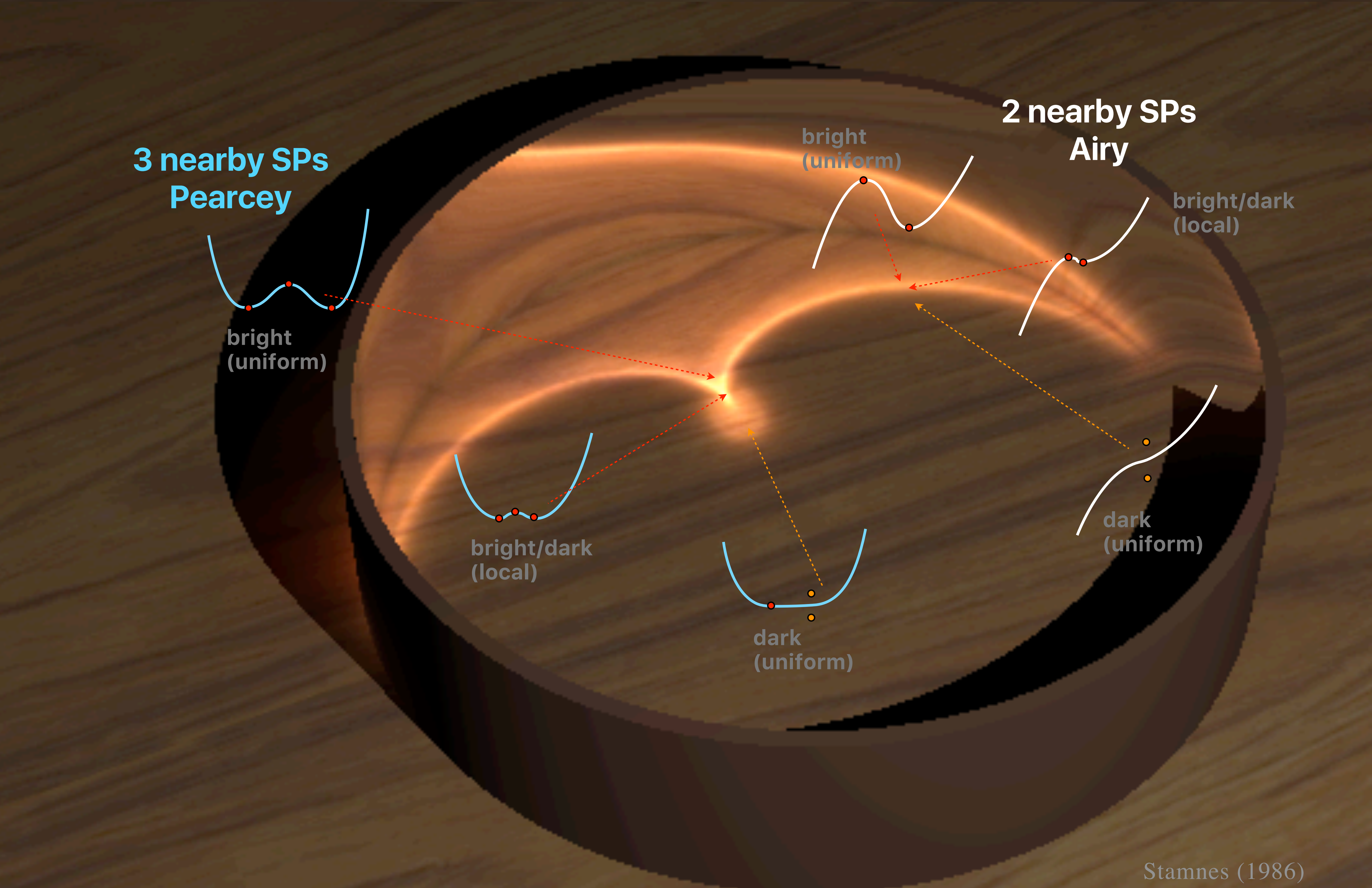
http://joshwills.com/rendering

Caustics



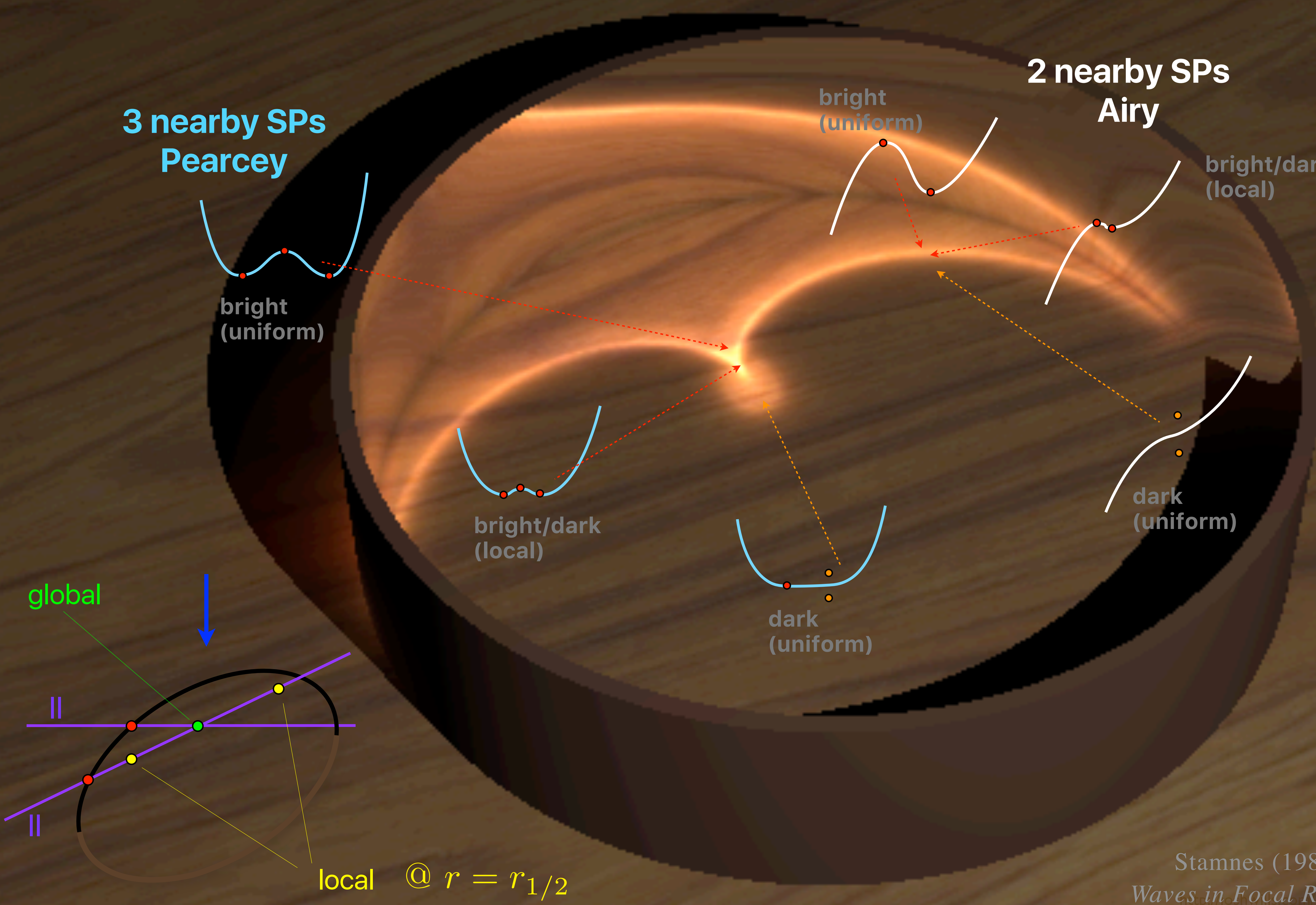
2 nearby SPs
Airy
bright
(uniform)
bright/dark
(local)
dark
(uniform)
http://joshwills.com/rendering

Caustics



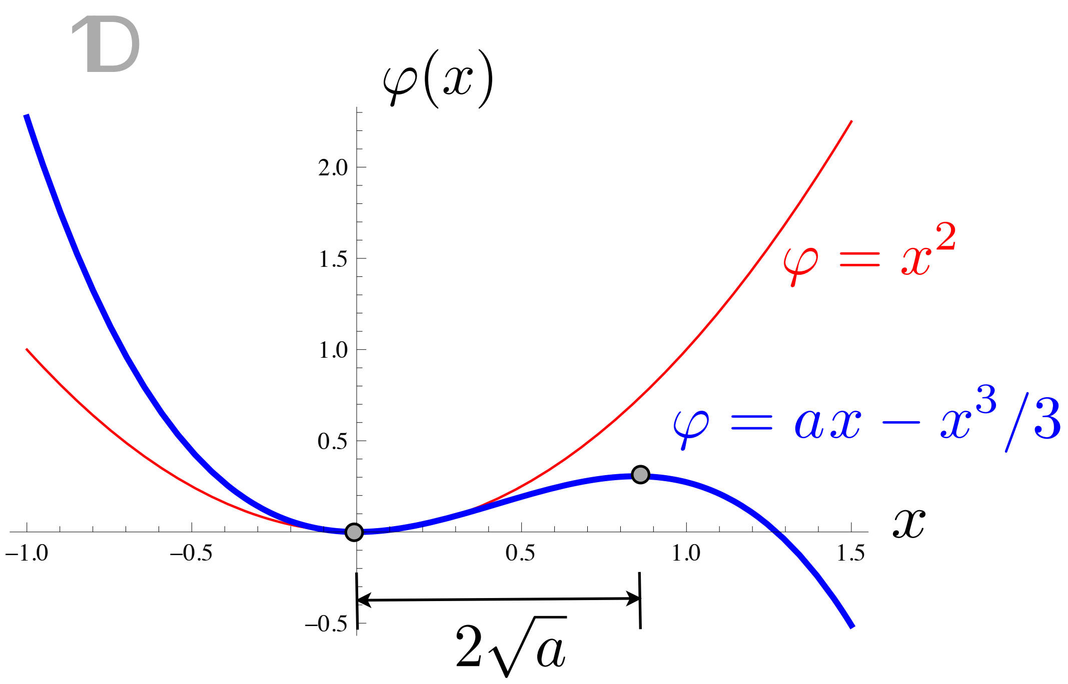
Stamnes (1986)
Waves in Focal Regions

Caustics



Stamnes (1986)
Waves in Focal Regions

Fold Catastrophe

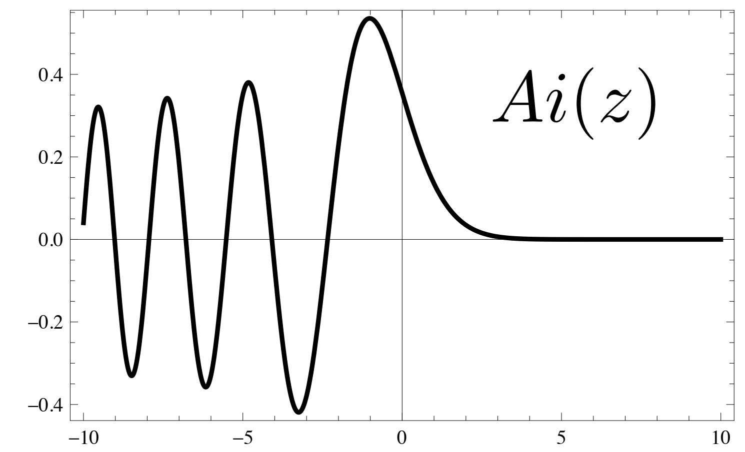


Model integral

Blestein & Handelsman (1986)

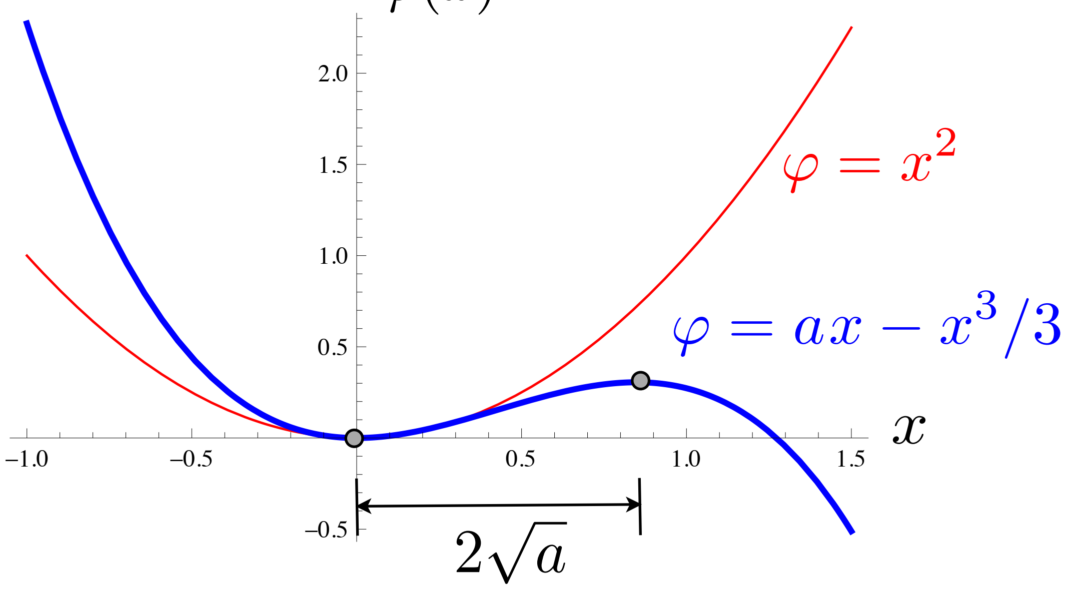
vs. $k^{-1/2}$

$$\int_{-\infty}^{\infty} e^{ik(\alpha x - x^3/3)}\, \mathrm{d}x \;=\; 2\pi\, k^{-1/3}\, Ai(-k^{2/3}\alpha)$$



Fold Catastrophe

1D



Model integral

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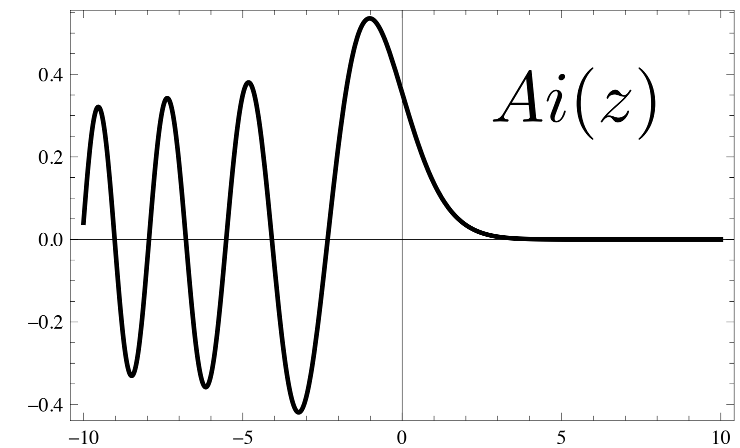
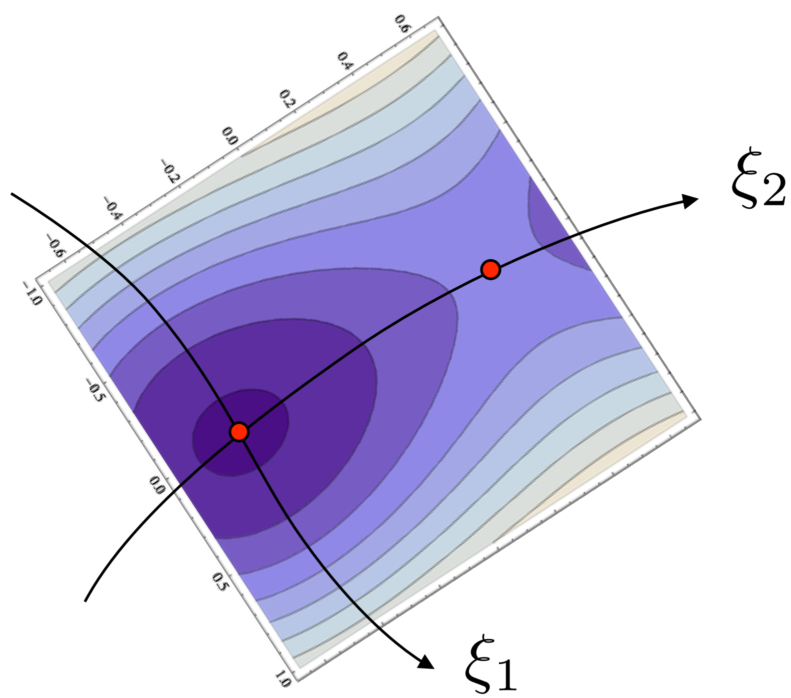
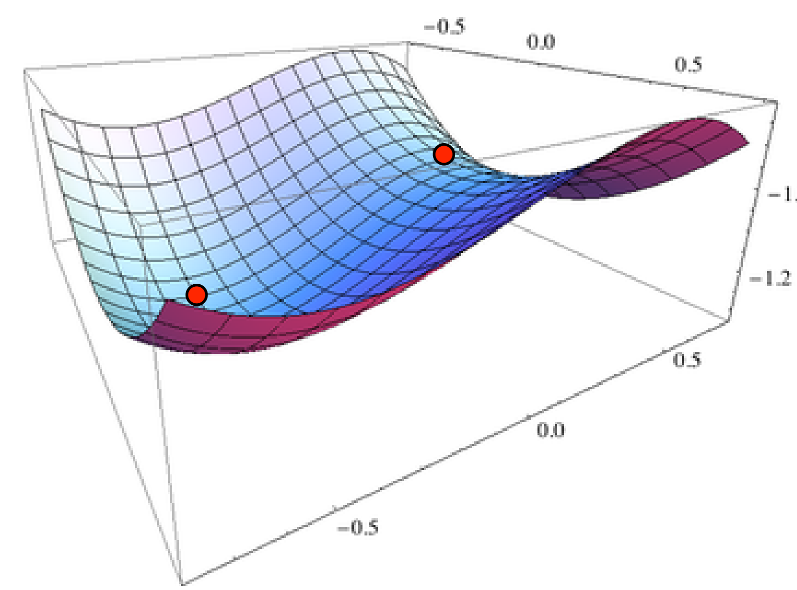


2D



Splitting lemma

$$\mathrm{rank}(\boldsymbol{H}) = 1$$



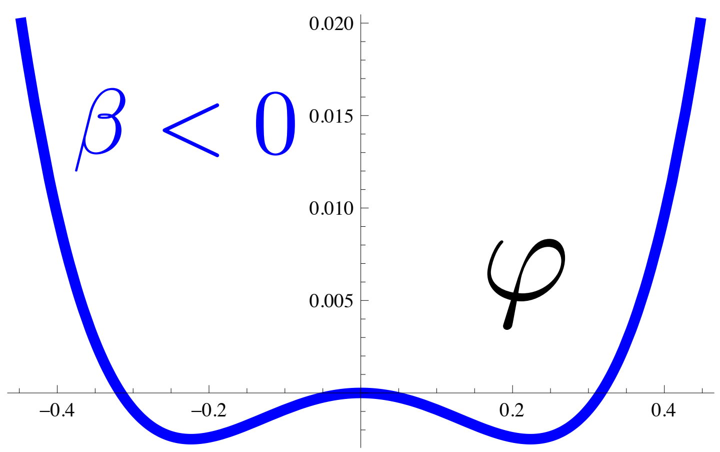
$$\varphi \;\Leftrightarrow\; \pm\,\xi_1^2 + \tilde{\phi}(\xi_2), \qquad a \geq 0$$

Cusp Catastrophe

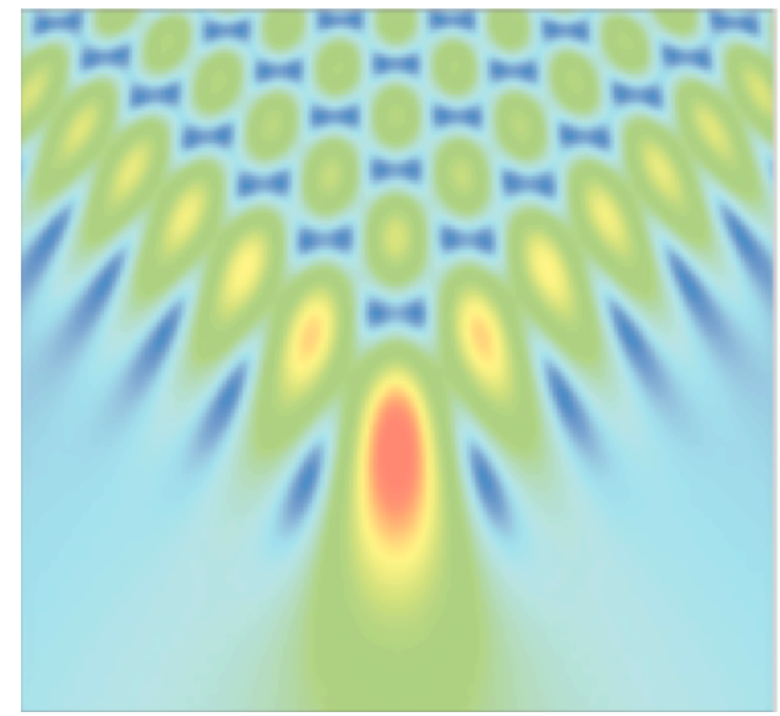
Trevor Pearcey (1919-1998)
CSIRO, Australia

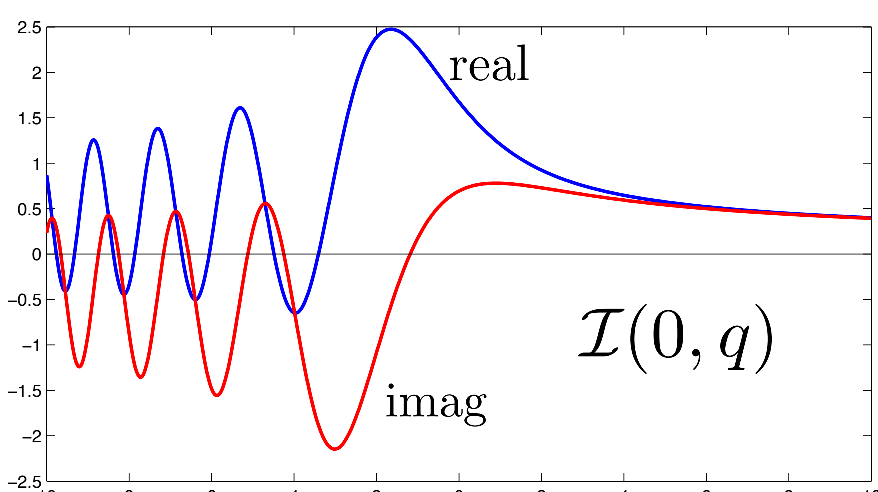
Canonical form (Pearcey integral) Brillouin (1916), Pearcey (1946)

$$\mathcal{I}(p,q) = \int_{-\infty}^{\infty} e^{\mathrm{i}(px+qx^2+x^4)}\,\mathrm{d}x$$



$$\varphi' = 0, \quad \varphi'', \varphi''' \to 0$$





$$\int_{-\infty}^{\infty} e^{\mathrm{i}k(\alpha x+\beta x^2+x^4)}\,\mathrm{d}x = k^{-1/4}\,\mathcal{I}(k^{3/4}\alpha,\, k^{1/2}\beta)$$

Borovikov (1994)

Connor & Farrelly (1973)

Near-caustic behavior

codimension

$$\mathrm{cod}(\phi) = \dim(H_2/\jmath\Delta(\phi))$$

Universal unfolding theorem

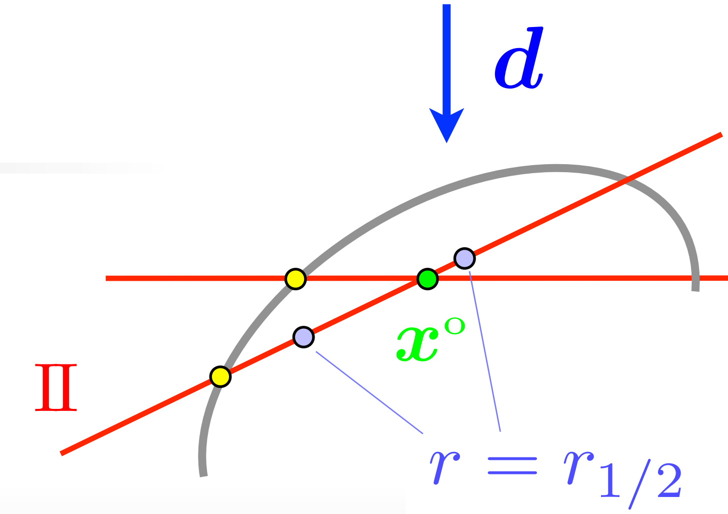
Uniform asymptotic treatment

germ

$$\phi(s,t) \;=\; \jmath\phi(s,t)\Big|_{(\boldsymbol{d},\boldsymbol{x}^\circ)\in B_\phi} + \sum_{m=1}^{M} c_m(\boldsymbol{d},\boldsymbol{x}^\circ)\, h_m(s,t), \qquad M = \mathrm{cod}(\phi)$$

Catastrophe	corank	cod	universal unfolding	μ	$\sigma_m^{\min}$	$\mathsf{T}^c(\boldsymbol{x}^\circ,\cdot,\cdot)$
Fold	1	1	$\pm s^2 + t^3/3 + ct$	1/6	2/3	$O(k^{7/6})$
Cusp	1	2	$\pm s^2 + t^4 + c_2 t^2 + c_1 t$	1/4	1/2	$O(k^{5/4})$
Swallowtail	1	3	$\pm s^2 + t^5 + c_3 t^3 + c_2 t^2 + c_1 t$	3/10	2/5	$O(k^{13/10})$
Hyp. umbilic	2	3	$s^3 + t^3 + c_3 st + c_2 t + c_1 s$	1/3	1/3	$O(k^{4/3})$
Ell. umbilic	2	3	$s^3 - st^2 + c_3(s^2 + t^2) + c_2 t + c_1 s$	1/3	1/3	$O(k^{4/3})$

Near-caustic behavior



codimension

$$\mathrm{cod}(\phi) = \dim(H_2 / \jmath\Delta(\phi))$$

Universal unfolding theorem

Uniform asymptotic treatment

germ

$$\phi(s,t) \;=\; \jmath\phi(s,t)\Big|_{(\boldsymbol{d},\boldsymbol{x}^\circ)\in B_\phi} + \sum_{m=1}^{M} c_m(\boldsymbol{d},\boldsymbol{x}^\circ)\, h_m(s,t), \qquad M = \mathrm{cod}(\phi)$$

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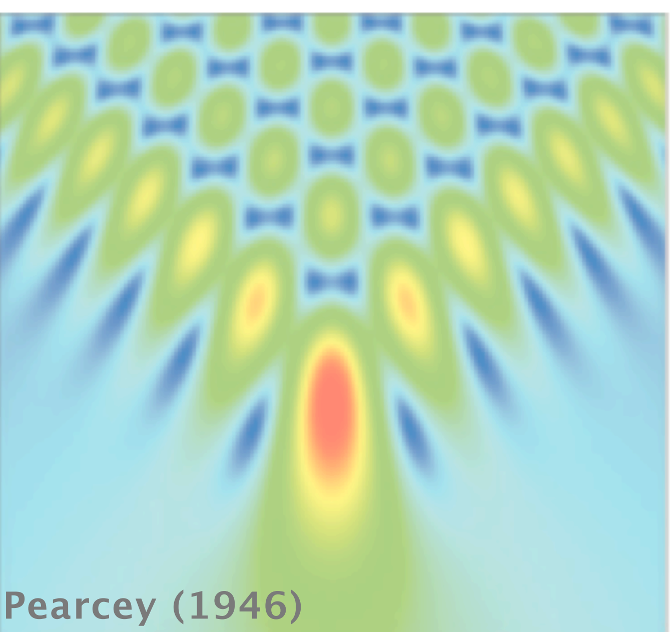
singularity index

$$\mathsf{T}^c(\boldsymbol{x}^\circ,\cdot,\cdot) \sim k^{\mu}\,\Psi(k^{\sigma_1}c_1,\ldots,k^{\sigma_M}c_M)$$

Diffraction scaling

canonical integral e.g. Airy (for Fold)

Stamnes (1986)
Waves in Focal Regions

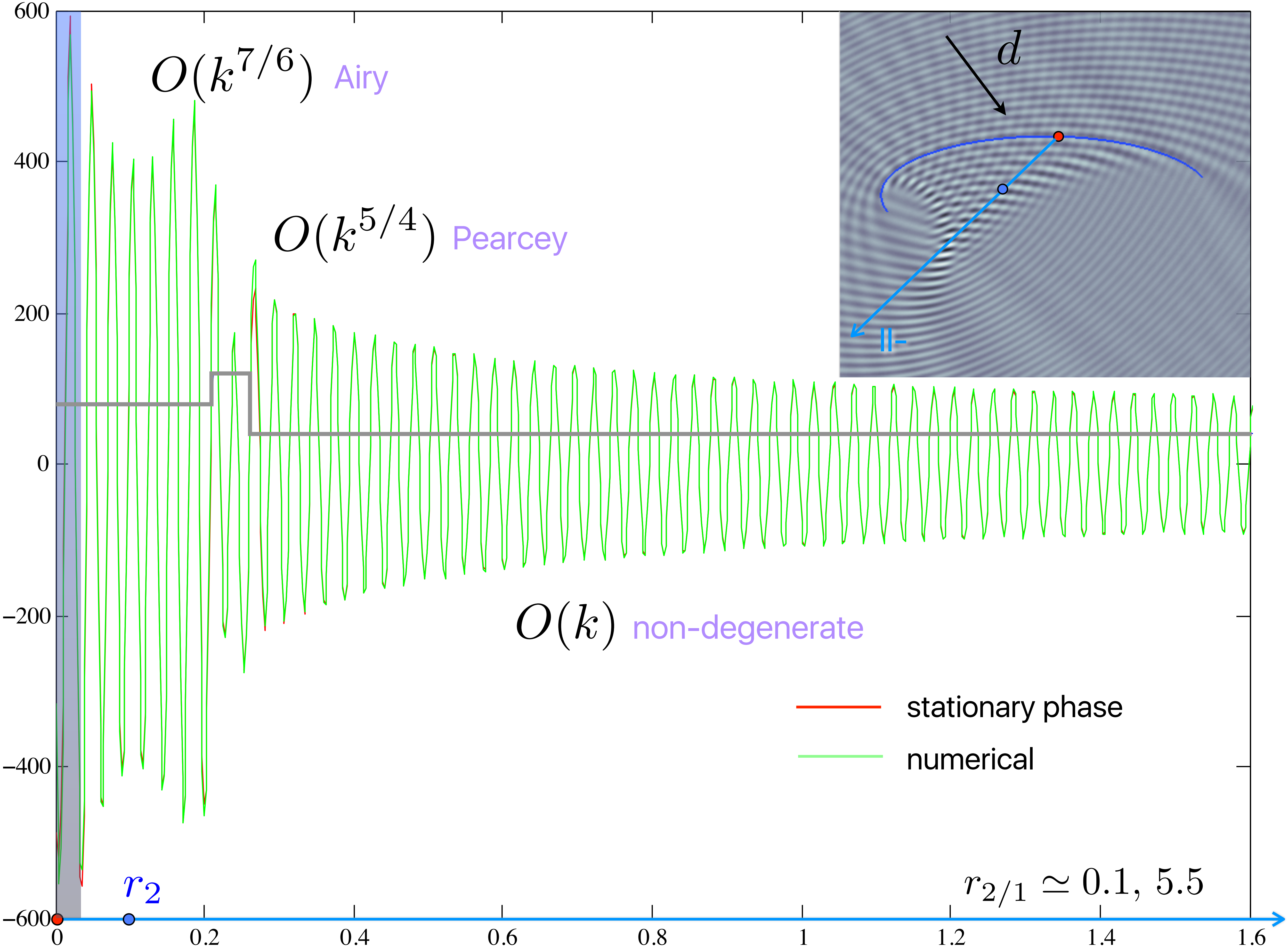


Pearcey (1946)

Pearcey integral

$$\int_{-\infty}^{\infty} e^{ik(\alpha x + \beta x^2 + x^4)}\,\mathrm{d}x \;=\; k^{-1/4}\,\mathcal{I}(k^{3/4}\alpha,\, k^{1/2}\beta)$$

Verification



Near-boundary behavior

$$J_1 = \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1+\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^{\circ}-\boldsymbol{\zeta})}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_{\zeta} + \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1-\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^{\circ}-\boldsymbol{\zeta})}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_{\zeta}$$

"slow" "fast"

$$\mathsf{T}(\boldsymbol{x}^{\circ}) = \int_{S^{\mathrm{f}}} f(\boldsymbol{\zeta})\,e^{\mathrm{i}k\varphi(\boldsymbol{\zeta})}\,\mathrm{d}\boldsymbol{\zeta}$$

Near-boundary behavior

$$J_1 = \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1+\frac{\mathrm{i}}{kr}\Big)\, \boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^\circ - \boldsymbol{\zeta})}\, e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta \;+\; \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1-\frac{\mathrm{i}}{kr}\Big)\, \boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^\circ - \boldsymbol{\zeta})}\, e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta$$

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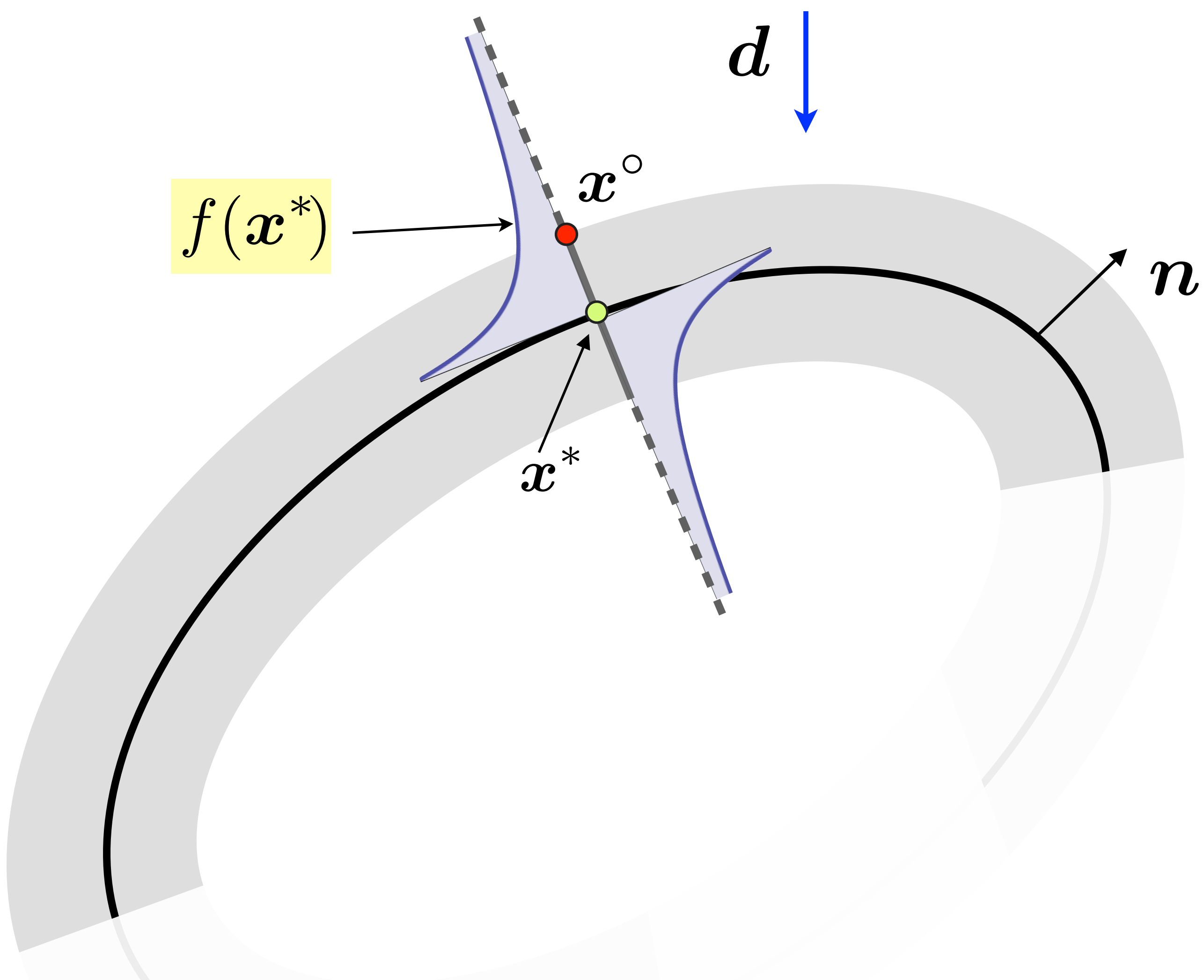

Near-boundary behavior

$$J_1 = \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\zeta)}{8\pi r}\left(1+\frac{\mathrm{i}}{kr}\right)\boldsymbol{d}\cdot(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})\, e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta + \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\zeta)}{8\pi r}\left(1-\frac{\mathrm{i}}{kr}\right)\boldsymbol{d}\cdot(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})\, e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta$$

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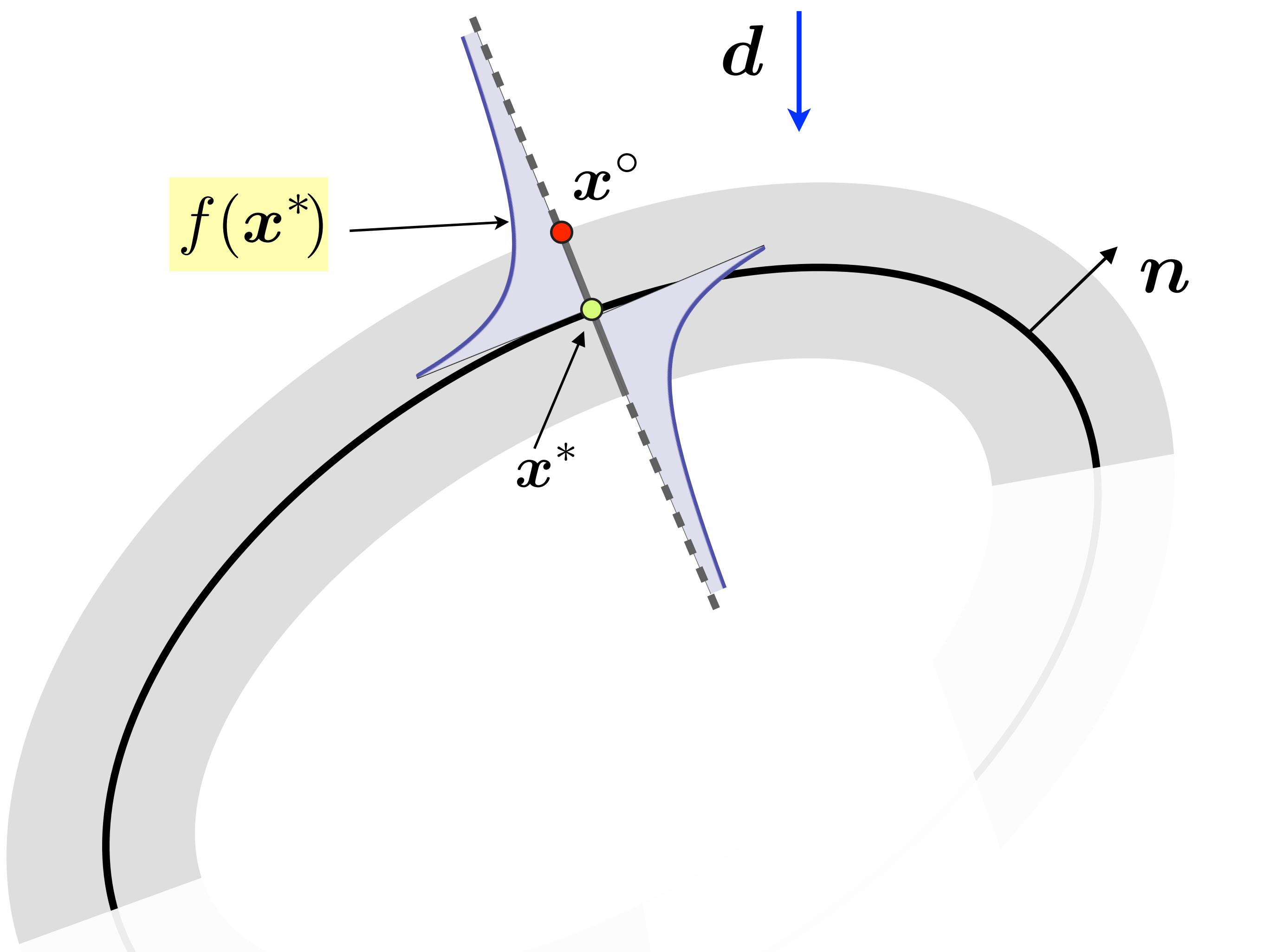
$$\mathsf{T}(\boldsymbol{x}^\circ) = \int_{S^{\mathrm{f}}} f(\boldsymbol{\zeta})\, e^{\mathrm{i}k\varphi(\boldsymbol{\zeta})}\,\mathrm{d}\boldsymbol{\zeta}$$

$\zeta_0 \in S^{\mathrm{f}}$ where f or φ fail to be differentiable $\Leftarrow$ $\boldsymbol{x}^\circ \in S^{\mathrm{f}}$



Near-boundary behavior

$$J_1 \;=\; \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\zeta)}{8\pi r}\left(1+\frac{\mathrm{i}}{kr}\right)\boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^{\circ}-\boldsymbol{\zeta})}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta \;+\; \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\zeta)}{8\pi r}\left(1-\frac{\mathrm{i}}{kr}\right)\boldsymbol{d}\cdot\widehat{(\boldsymbol{x}^{\circ}-\boldsymbol{\zeta})}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta$$

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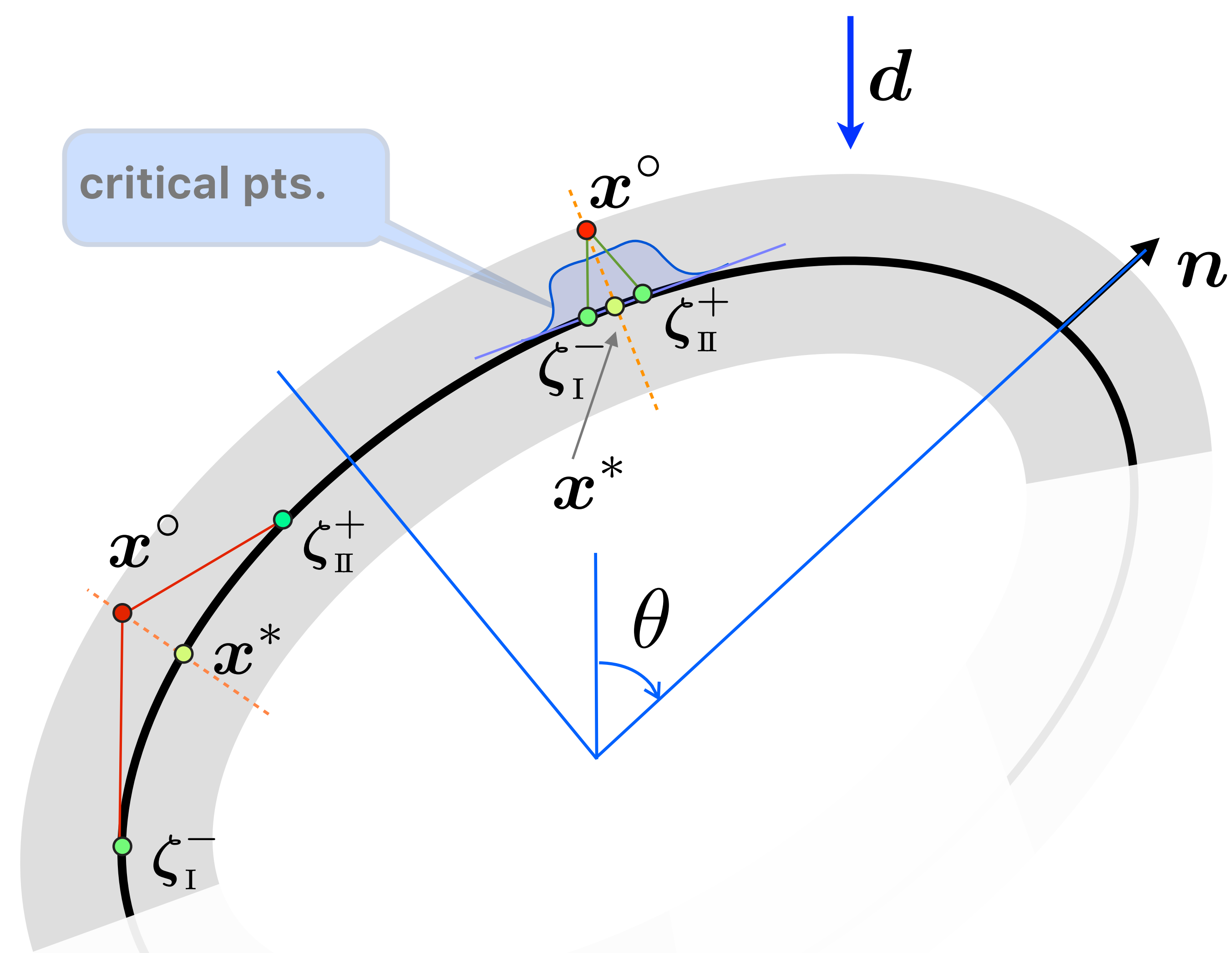
$\zeta_0 \in S^{\mathrm{f}}$ where f or φ fail to be differentiable $\Leftarrow$ $\boldsymbol{x}^{\circ} \in S^{\mathrm{f}}$

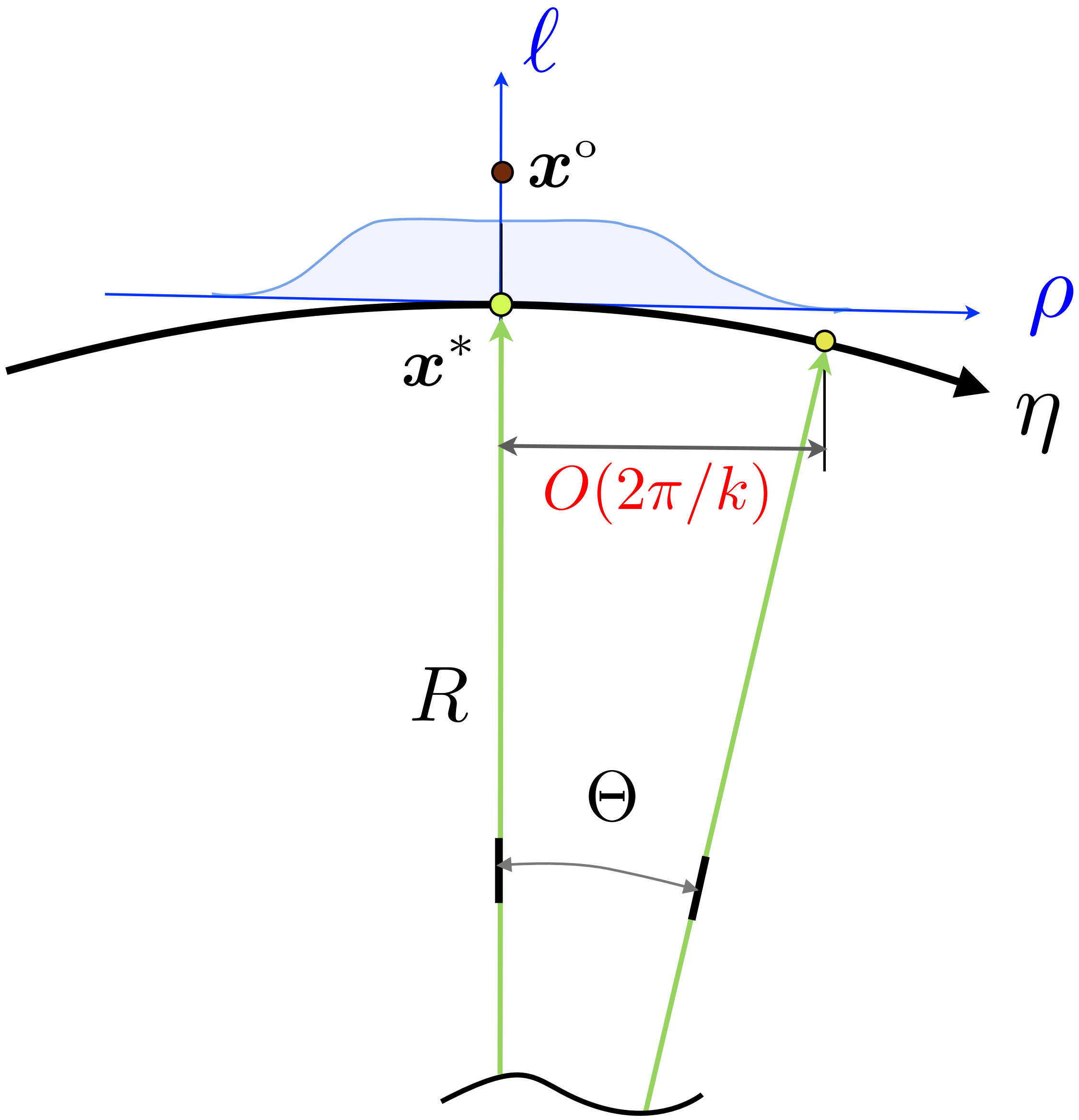
Normal distance

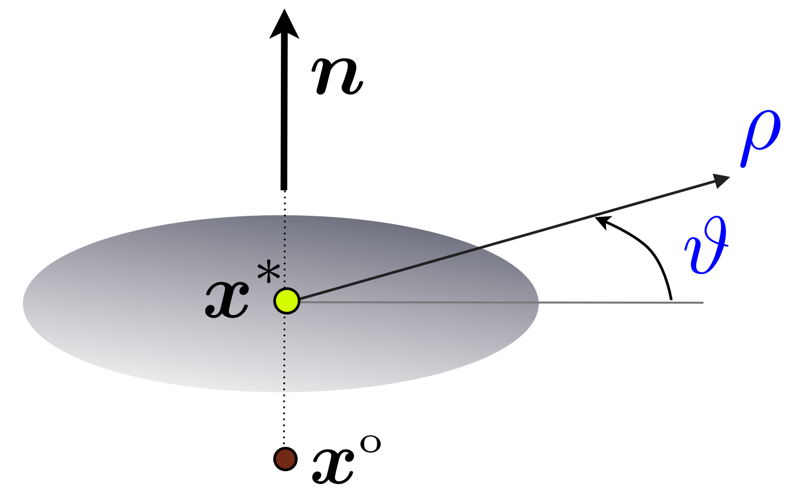
$$\ell \;=\; |\boldsymbol{x}^{\circ} - \boldsymbol{x}^{*}|$$

$$\ell \;\leqslant\; \frac{2\pi}{k}$$



Planar surface approximation

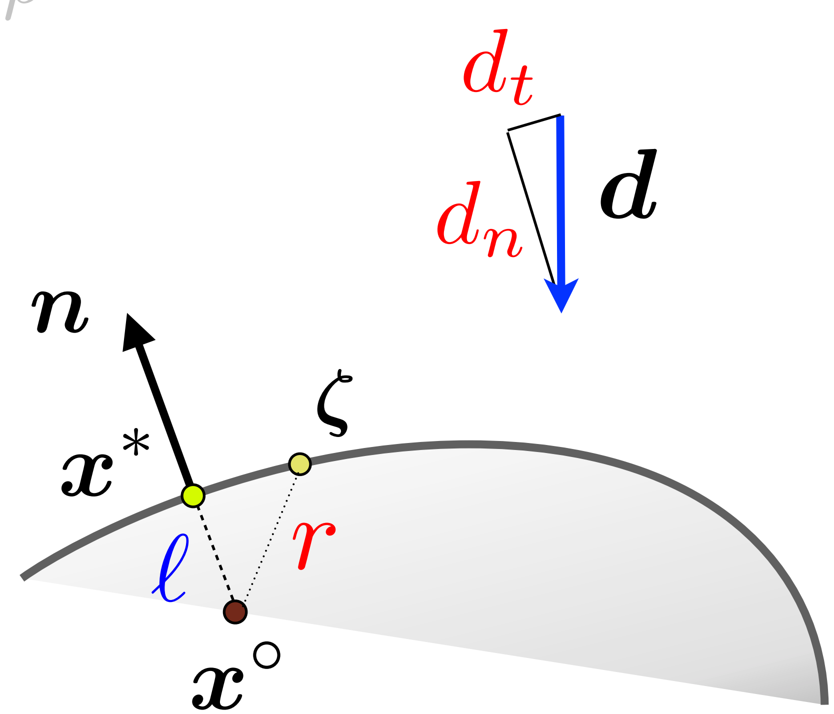




$$d_n = |\boldsymbol{d}\cdot\boldsymbol{n}|$$

$$d_t = \sqrt{1 - |\boldsymbol{d}\cdot\boldsymbol{n}|^2}$$

$$r = \sqrt{\ell^2 + \rho^2}$$



$$\Theta = \eta/R = O(2\pi/k)$$

$$\rho = R\sin(\Theta) = R(\Theta - \Theta^3/6 + \ldots)$$

$$= \eta\,[1 + O(2\pi/k^2)]$$

Evaluation

$$\frac{\sin(kr)}{kr}$$

Evaluation

$$\frac{\sin(kr)}{kr}$$

$\mathsf{T} = \mathsf{T}^* + \mathsf{T}^s \rightarrow J_1 = J_1^* + J_1^s\,, \quad J_2 = J_2^* + J_2^s$

Evaluation

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$\mathsf{T} = \mathsf{T}^* + \mathsf{T}^{\mathsf{s}} \rightarrow J_1 = J_1^* + J_1^s\,, \quad J_2 = J_2^* + J_2^s$

$$J_1 \;=\; \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1+\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta \;+\; \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1-\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta,$$

$$J_2 \;=\; \mathrm{i}\int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta \;-\; \mathrm{i}\int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta, \qquad r = |\boldsymbol{x}^\circ-\boldsymbol{\zeta}|, \qquad \boldsymbol{x}^\circ \notin S^{\mathrm{f}}.$$

Evaluation

$$\frac{\sin(kr)}{kr}$$

$$\mathsf{T} = \mathsf{T}^* + \mathsf{T}^{\mathsf{s}} \rightarrow \quad J_1 = J_1^* + J_1^s, \quad J_2 = J_2^* + J_2^s$$

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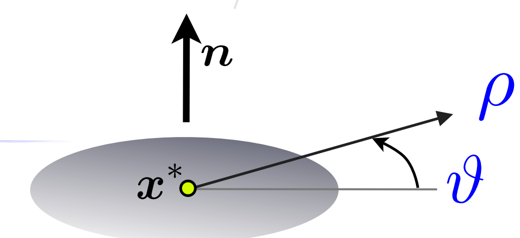
$$J_2 = \mathrm{i}\int_{S^{\rm f}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r} e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta - \mathrm{i}\int_{S^{\rm f}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r} e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta, \qquad r = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}|, \qquad \boldsymbol{x}^\circ \notin S^{\rm f}.$$



Polar coordinates

$$J_1^* = \frac{d_n}{4\pi} e^{\mathrm{i}k\boldsymbol{x}^*\cdot\,\boldsymbol{d}} \int_0^\infty \frac{k\rho}{(kr)^2}\left[\cos(kr) - \frac{\sin(kr)}{kr}\right] \int_0^{2\pi} \Big(\pm\, d_n k\ell + d_t \cos(\vartheta)\, k\rho\Big)\, e^{\mathrm{i}d_t k\rho\cos(\vartheta)}\, d\vartheta\, d\rho$$

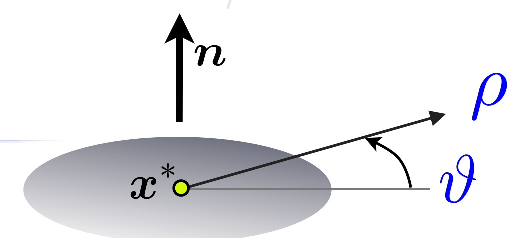
Evaluation

$\frac{\sin(kr)}{kr}$

$$\mathsf{T} = \mathsf{T}^* + \mathsf{T}^{\mathsf{s}} \quad \rightarrow \quad J_1 = J_1^* + J_1^s\,, \qquad J_2 = J_2^* + J_2^s$$

$$J_1 \;=\; \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1+\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta \;+\; \int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\Big(1-\frac{\mathrm{i}}{kr}\Big)\,\boldsymbol{d}\cdot(\widehat{\boldsymbol{x}^\circ-\boldsymbol{\zeta}})\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta$$

$$J_2 \;=\; \mathrm{i}\int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}+r)}\,\mathrm{d}S_\zeta \;-\; \mathrm{i}\int_{S^{\mathrm{f}}} \frac{\boldsymbol{d}\cdot\boldsymbol{n}(\boldsymbol{\zeta})}{8\pi r}\,e^{\mathrm{i}k(\boldsymbol{\zeta}\cdot\boldsymbol{d}-r)}\,\mathrm{d}S_\zeta, \qquad r=|\boldsymbol{x}^\circ-\boldsymbol{\zeta}|, \qquad \boldsymbol{x}^\circ\notin S^{\mathrm{f}}.$$



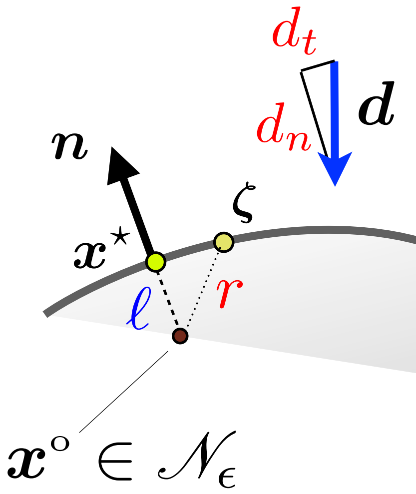
Polar coordinates

$$J_1^* = \frac{d_n}{4\pi}e^{\mathrm{i}k\boldsymbol{x}^*\cdot\,\boldsymbol{d}}\int_0^\infty \frac{k\rho}{(kr)^2}\Big[\cos(kr)-\frac{\sin(kr)}{kr}\Big]\int_0^{2\pi}\Big(\pm\, d_n k\ell + d_t\cos(\vartheta)\,k\rho\Big)\,e^{\mathrm{i}d_t k\rho\cos(\vartheta)}\,d\vartheta\,d\rho$$

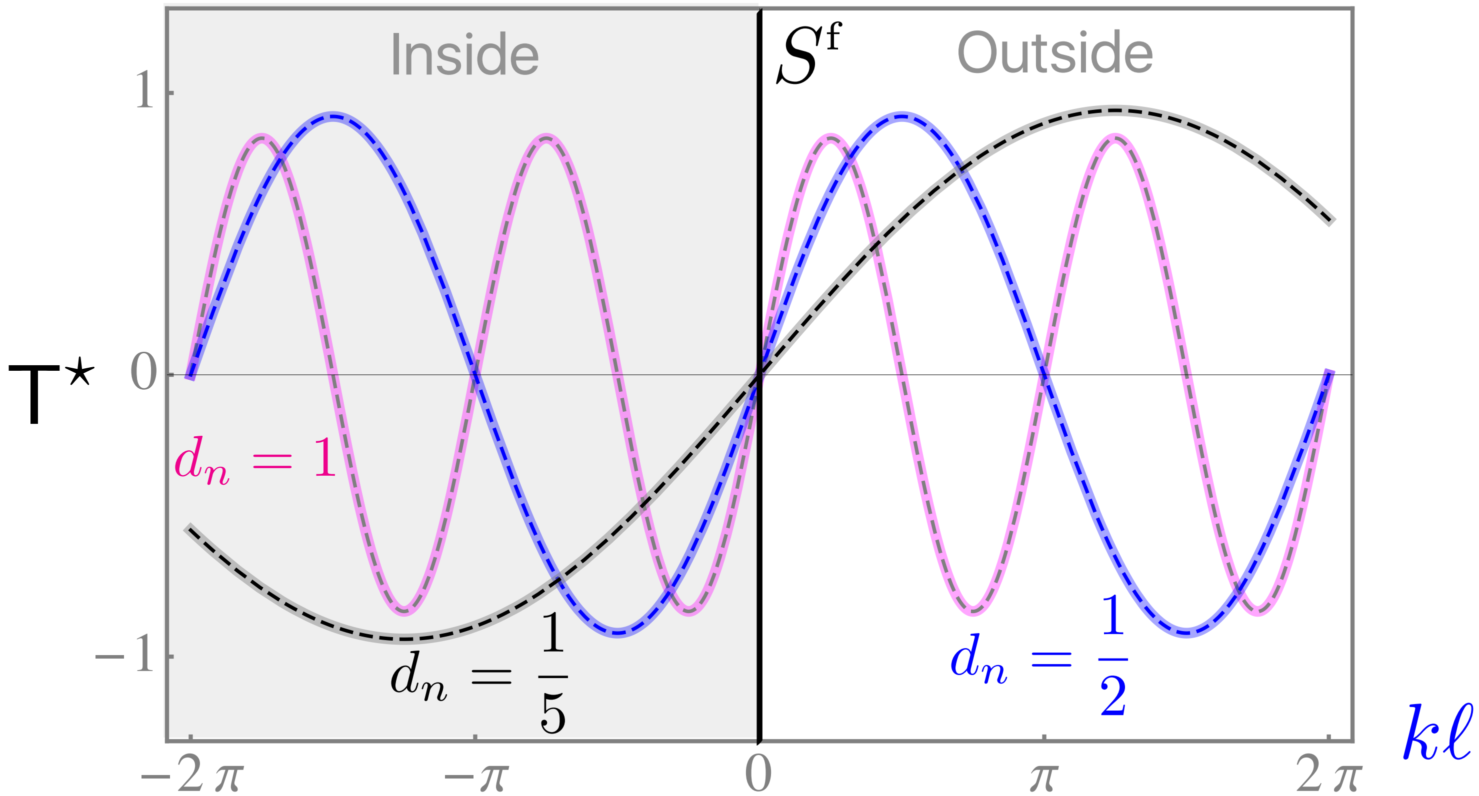
$$J_1^* = \frac{d_n}{2k}\,e^{\mathrm{i}k\boldsymbol{x}^*\cdot\,\boldsymbol{d}}\int_0^\infty \frac{k\rho}{(kr)^2}\Big[\cos(kr)-\frac{\sin(kr)}{kr}\Big]\Big(\pm\, d_n k\ell\,\mathcal{J}_0(d_t k\rho) + \mathrm{i}\,d_t k\rho\,\mathcal{J}_1(d_t k\rho)\Big)\,d(k\rho)$$

$$J_2 = \frac{d_n}{2}\,e^{\mathrm{i}k\boldsymbol{x}^*\boldsymbol{d}}\int_0^\infty \frac{k\rho}{kr}\sin(kr)\,\mathcal{J}_0(d_t k\rho)\,d\rho \qquad\qquad r=\sqrt{\ell^2+\rho^2}$$

$x^\star$ contribution



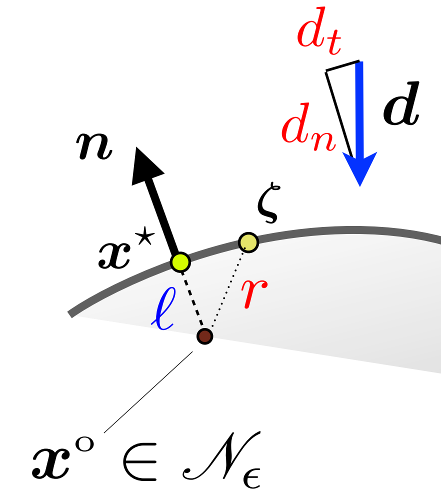
$$\mathsf{T}^\star(k\ell, d_n; \beta, \gamma) = \frac{k}{2} \sin(2k\ell d_n) \left\{ \frac{3(1-\beta)}{2+\beta} (2d_n{}^2 - 1) - (1 - \beta\gamma^2) \right\}$$



distant

nearby

$\boldsymbol{x}^\star$ contribution



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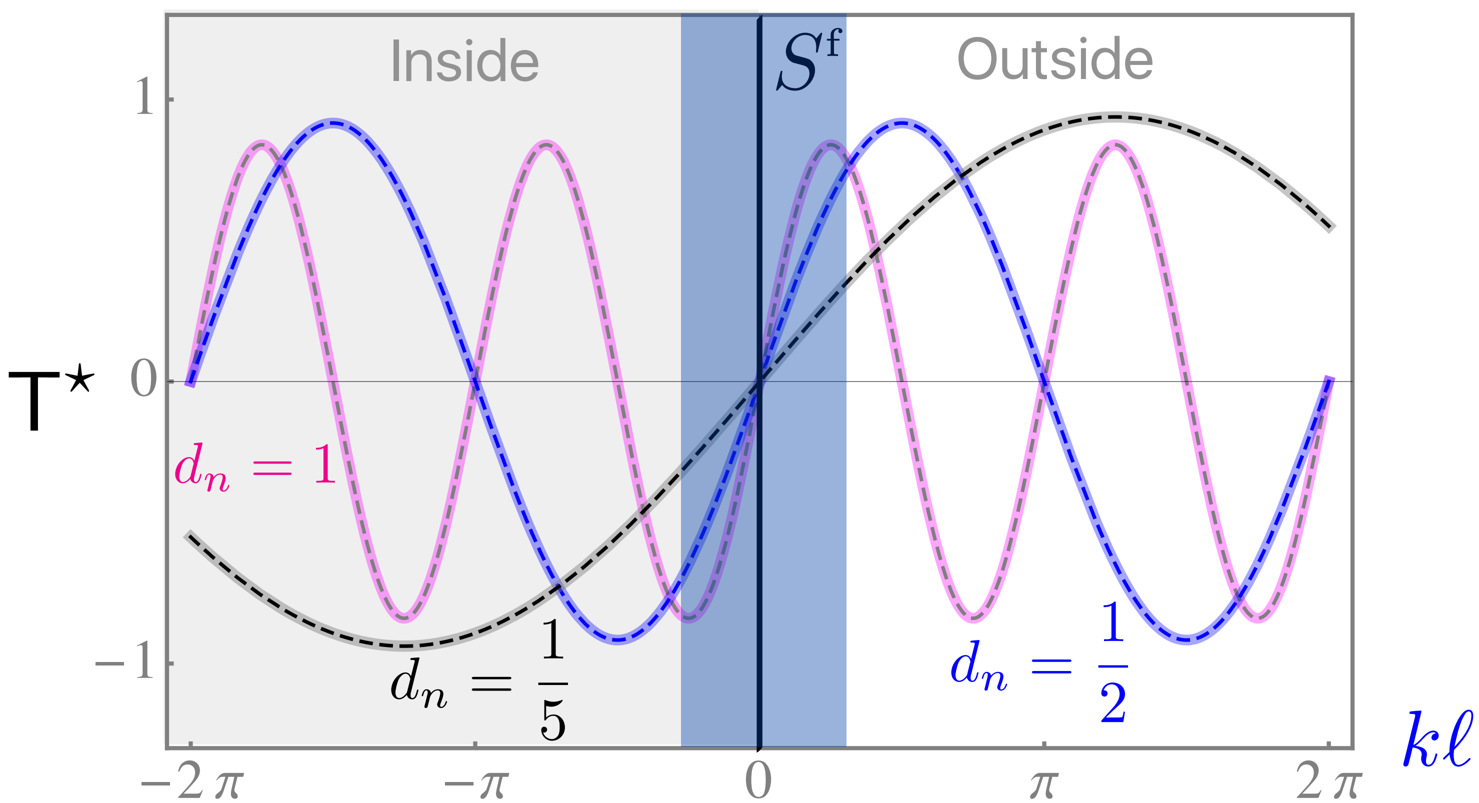


As $\boldsymbol{x}^\circ \to S^{\mathrm{f}}$

$$\mathsf{T}^\star = O(k)$$

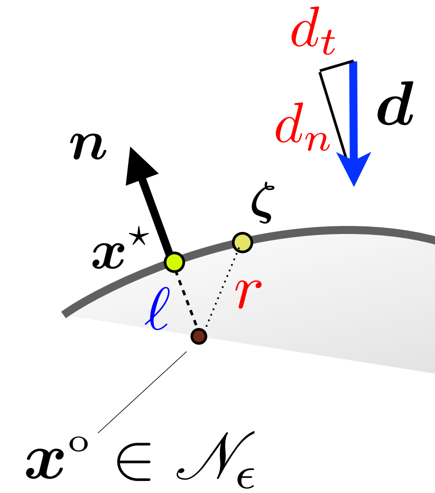
Apparent wavenumber along $\Pi^\pm$

$2k(\boldsymbol{d}\cdot\boldsymbol{n})^2$ distant

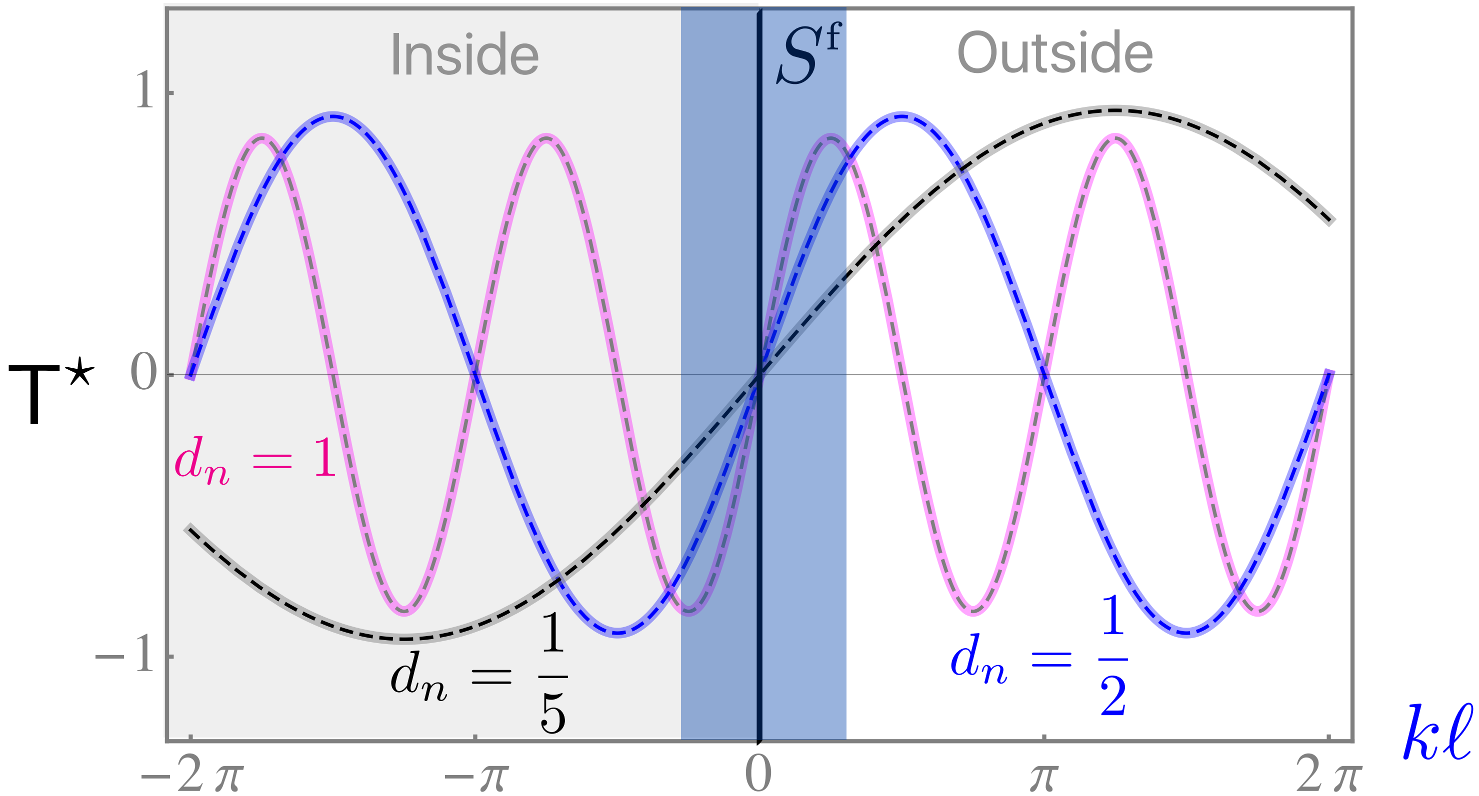
v.s.

nearby $2k(|\boldsymbol{d}\cdot\boldsymbol{n}|)$

$x^\star$ contribution



$$\mathsf{T}^\star(k\ell, d_n; \beta, \gamma) = \frac{k}{2}\sin(2k\ell d_n)\left\{\frac{3(1-\beta)}{2+\beta}(2{d_n}^2 - 1) - (1-\beta\gamma^2)\right\}$$



As $\boldsymbol{x}^\circ \to S^{\mathrm{f}}$

$$\mathsf{T}^\star = O(k)$$

Apparent wavenumber along $\mathrm{II}^\pm$

$2k(\boldsymbol{d}\cdot\boldsymbol{n})^2$ distant

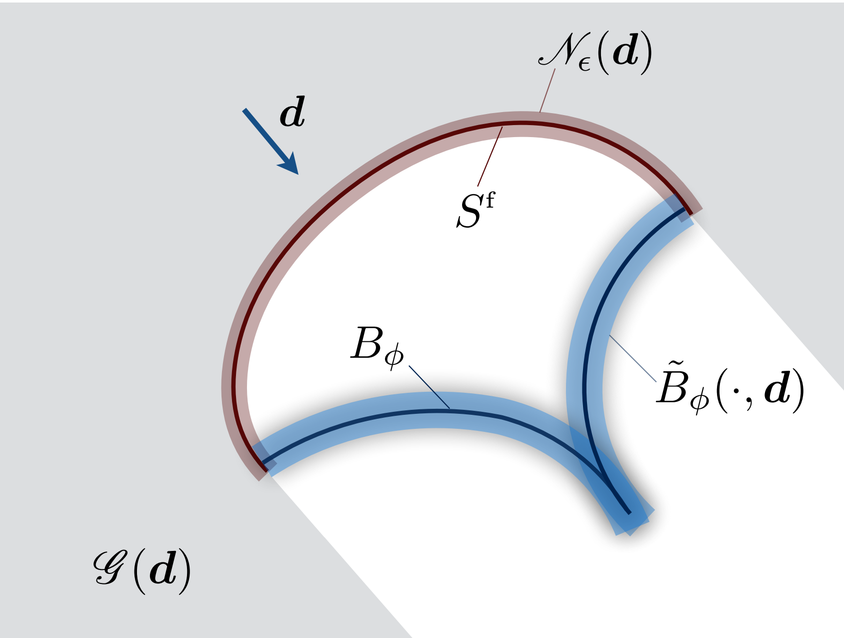
v.s.

nearby $2k(|\boldsymbol{d}\cdot\boldsymbol{n}|)$

Single plane-wave incident

$$\mathsf{T}(\boldsymbol{x}^\circ, \beta, \gamma) \overset{k^\nu}{=} 1_{\mathscr{N}_\epsilon(\boldsymbol{d})}(\boldsymbol{x}^\circ)\,\mathsf{T}^\star + 1_{\tilde{B}_\phi}(\boldsymbol{d}, \boldsymbol{x}^\circ)\,\mathsf{T}^c + 1_{\mathscr{G}(\boldsymbol{d})}(\boldsymbol{x}^\circ)\,\mathsf{T}^{\mathrm{II}^+} + \sum \mathsf{T}^{\mathrm{II}^-}$$

$\nu \leqslant \frac{1}{2}$



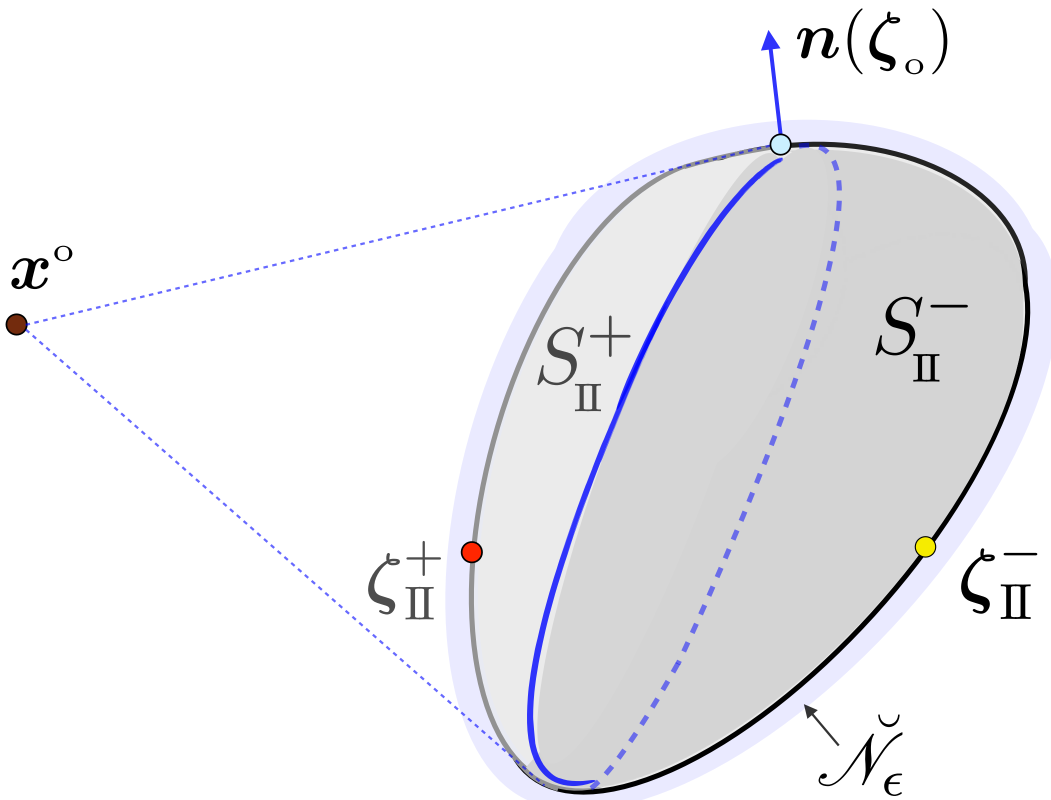
$$\mathsf{T}^\star = O(k), \qquad \mathsf{T}^c = O(k^\alpha),\ \ \frac{7}{6} \leqslant \alpha \leqslant \frac{4}{3}, \qquad \mathsf{T}^{\mathrm{II}^\pm} = O(k)$$

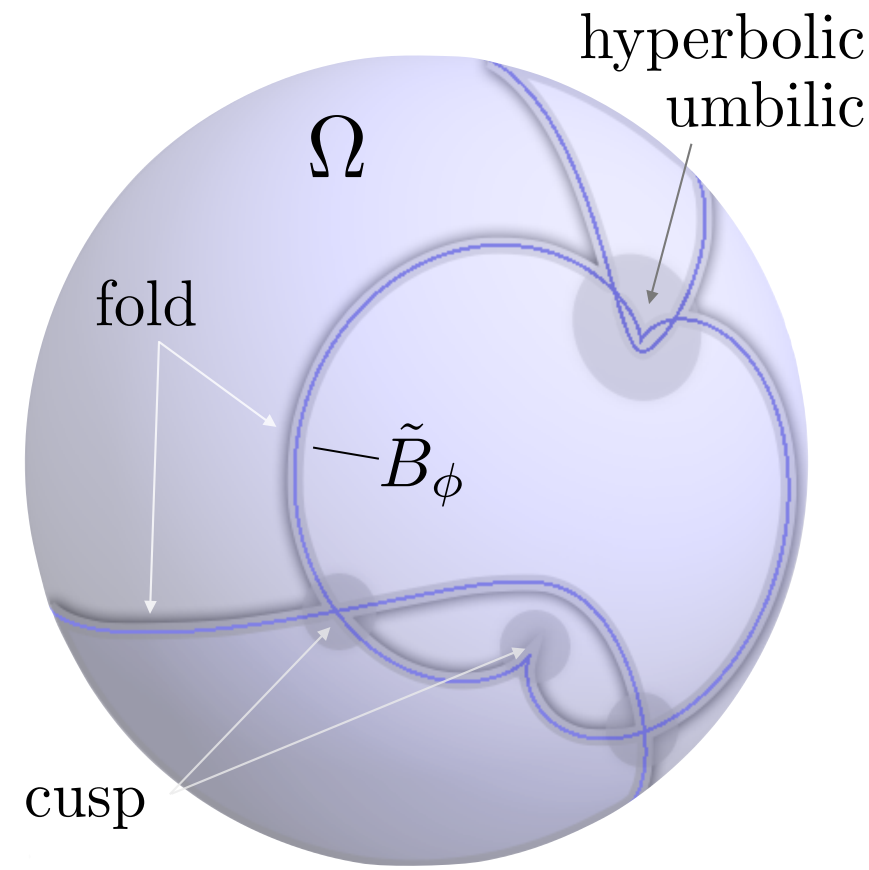
Full aperture "non-degenerate SPs"

$$\breve{\mathsf{T}}(\boldsymbol{x}^\circ,\beta,\gamma) \;=\; \int_\Omega \mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma)\,\mathrm{d}\Omega_{\boldsymbol{d}}$$

$$\breve{\mathsf{T}} \overset{k^\nu}{=} 1_{\breve{\mathcal{N}}_\epsilon}(\boldsymbol{x}^\circ)\int_\Omega \mathsf{T}^\star\;\mathrm{d}\Omega_{\boldsymbol{d}} + \int_{\tilde{B}_\phi} \mathsf{T}^c\;\mathrm{d}\Omega_{\boldsymbol{d}} + \int_{S^\pm} \mathsf{T}^{\mathrm{II}^\pm}\,\frac{|\boldsymbol{d}^*\!\cdot\boldsymbol{n}|}{r^2}\,\mathrm{d}S_\zeta$$

$$|\boldsymbol{d}^*\!\cdot\boldsymbol{n}| = |(\widehat{\boldsymbol{\zeta}-\boldsymbol{x}^\circ})\cdot\boldsymbol{n}(\boldsymbol{\zeta})|$$
$$r = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}|$$





Full aperture

"non-degenerate SPs"

$$\breve{\mathsf{T}}(\boldsymbol{x}^\circ,\beta,\gamma) \;=\; \int_\Omega \mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma)\,\mathrm{d}\Omega_{\boldsymbol{d}}$$

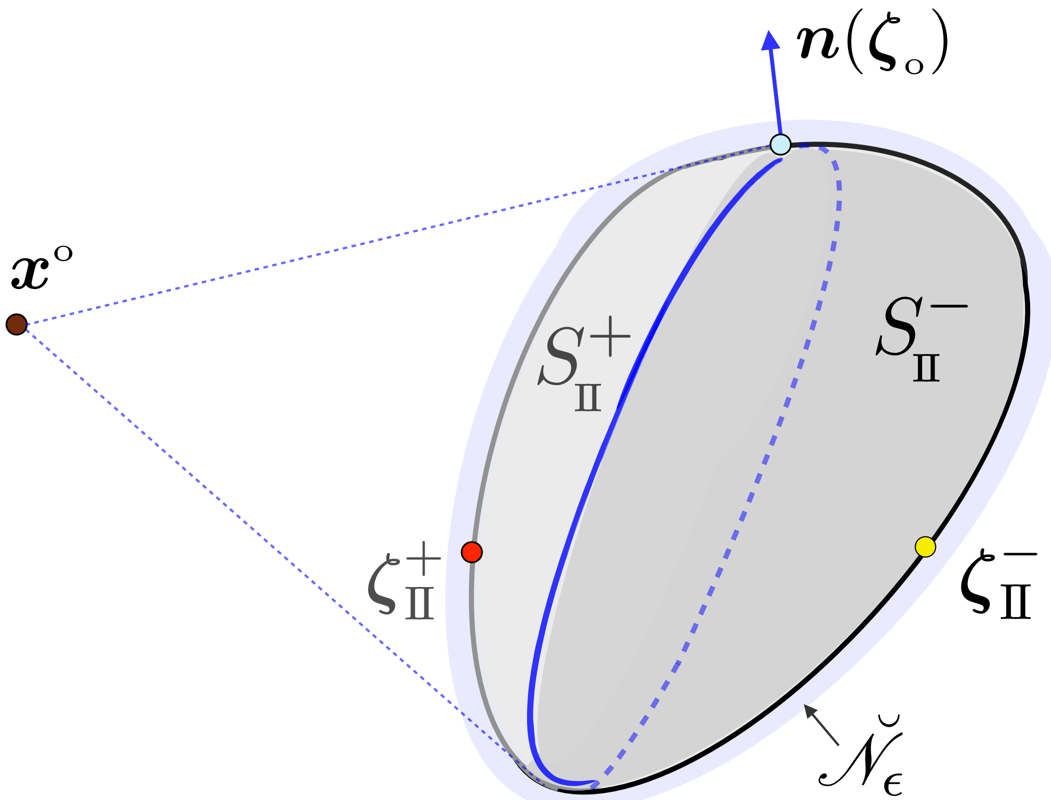
$$|\boldsymbol{d}^*\!\cdot\boldsymbol{n}| = |(\widehat{\zeta - \boldsymbol{x}^\circ})\cdot\boldsymbol{n}(\zeta)|$$

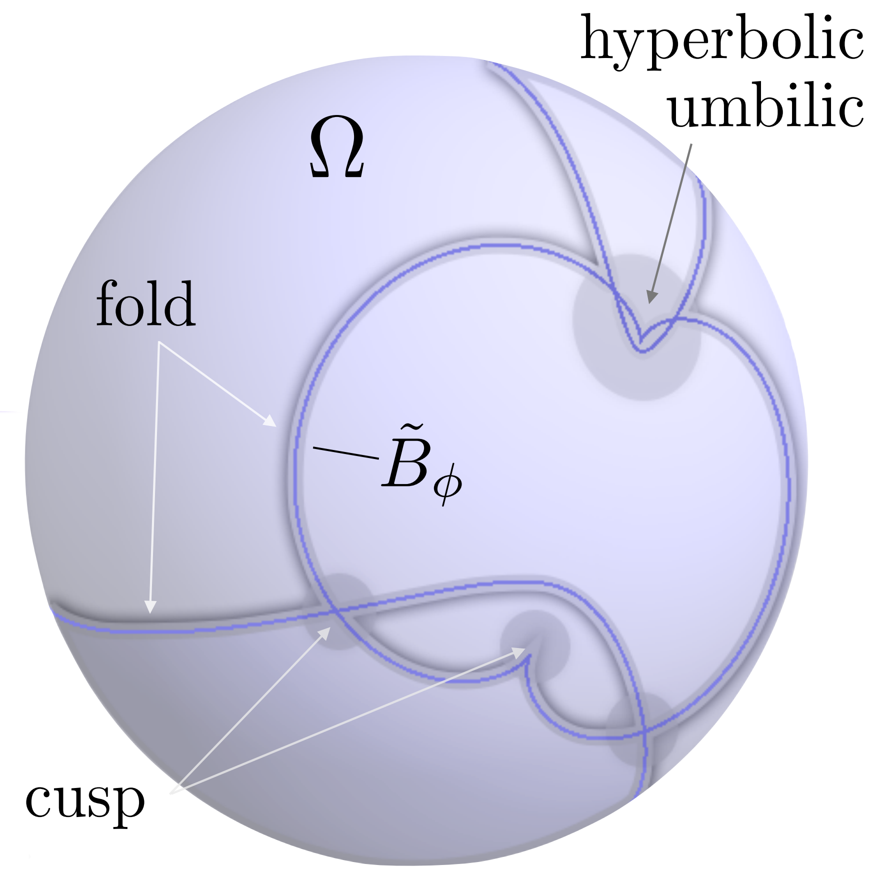
$$r = |\boldsymbol{x}^\circ - \zeta|$$

$$\breve{\mathsf{T}} \overset{k^\nu}{=} 1_{\breve{\mathscr{N}}_\epsilon}(\boldsymbol{x}^\circ)\int_\Omega \mathsf{T}^\star\;\mathrm{d}\Omega_{\boldsymbol{d}} \;+\; \int_{\tilde{B}_\phi} \mathsf{T}^c\;\mathrm{d}\Omega_{\boldsymbol{d}} \;+\; \int_{S^\pm} \mathsf{T}^{\mathrm{II}^\pm}\,\frac{|\boldsymbol{d}^*\!\cdot\boldsymbol{n}|}{r^2}\,\mathrm{d}S_\zeta$$

non-degenerate SPs

$$\boldsymbol{d}(\boldsymbol{x}^\circ,\zeta) \;=\; \pm[\boldsymbol{I} - 2\,\boldsymbol{n}\otimes\boldsymbol{n}(\zeta)]\,(\widehat{\zeta - \boldsymbol{x}^\circ})$$





Full aperture

"non-degenerate SPs"

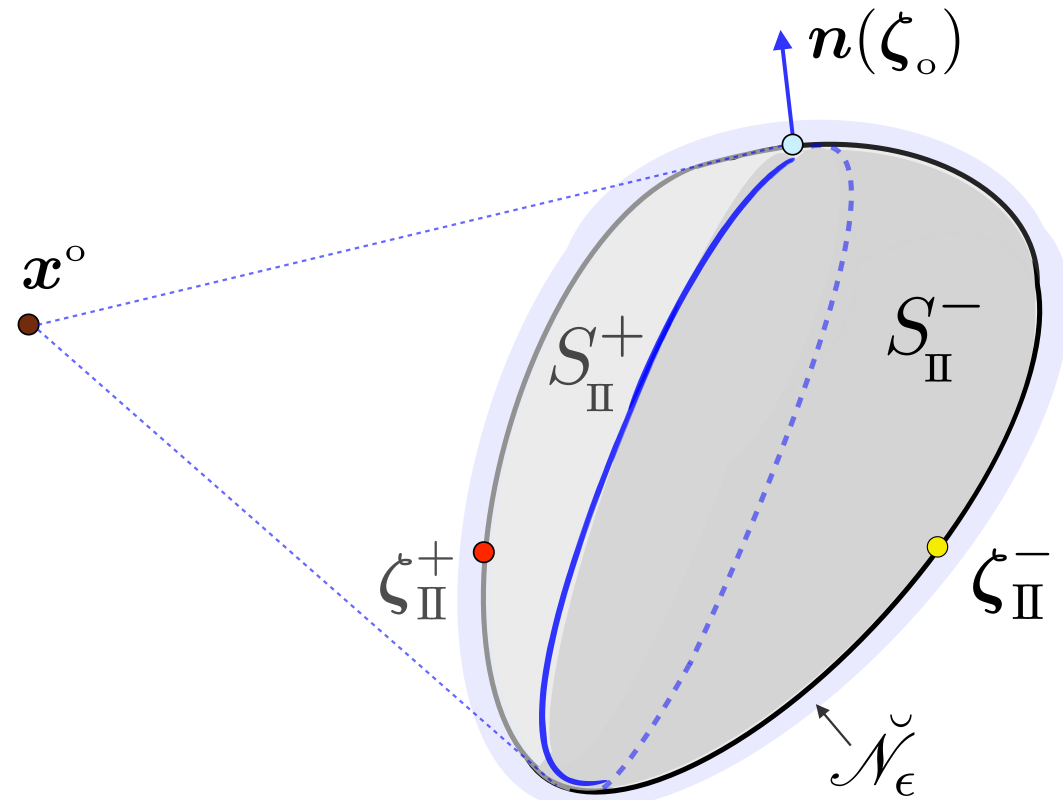
$$\breve{\mathsf{T}}(\boldsymbol{x}^\circ,\beta,\gamma) \;=\; \int_\Omega \mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma)\,\mathrm{d}\Omega_{\boldsymbol{d}}$$

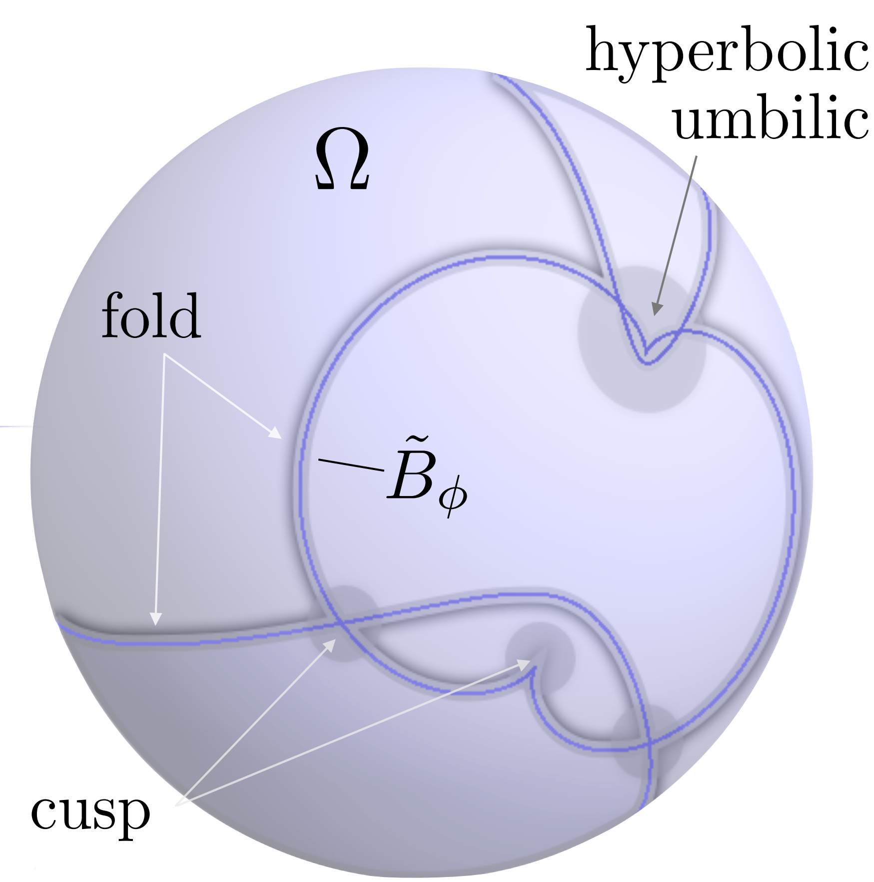
$$|\boldsymbol{d}^*\!\cdot\boldsymbol{n}| = |(\widehat{\boldsymbol{\zeta}-\boldsymbol{x}^\circ})\cdot\boldsymbol{n}(\boldsymbol{\zeta})|$$
$$r = |\boldsymbol{x}^\circ - \boldsymbol{\zeta}|$$

$$\breve{\mathsf{T}} \overset{k^\nu}{=} 1_{\breve{\mathscr{N}}_\epsilon}(\boldsymbol{x}^\circ)\int_\Omega \mathsf{T}^\star\,\mathrm{d}\Omega_{\boldsymbol{d}} + \int_{\tilde{B}_\phi}\mathsf{T}^c\,\mathrm{d}\Omega_{\boldsymbol{d}} + \int_{S^\pm}\mathsf{T}^{\mathrm{II}^\pm}\frac{|\boldsymbol{d}^*\!\cdot\boldsymbol{n}|}{r^2}\,\mathrm{d}S_\zeta$$

non-degenerate SPs

$$1_{\mathscr{G}(\boldsymbol{d})}(\boldsymbol{x}^\circ)\int_\Omega \mathsf{T}^{\mathrm{II}^+}\,\mathrm{d}\Omega_{\boldsymbol{d}} \;=\; -\int_{S^+_{\mathrm{II}}}\mathsf{T}^{\mathrm{II}^+}\frac{|\boldsymbol{d}^*\!\cdot\boldsymbol{n}|}{r^2}\,\mathrm{d}S_\zeta$$





Full aperture

"non-degenerate SPs"

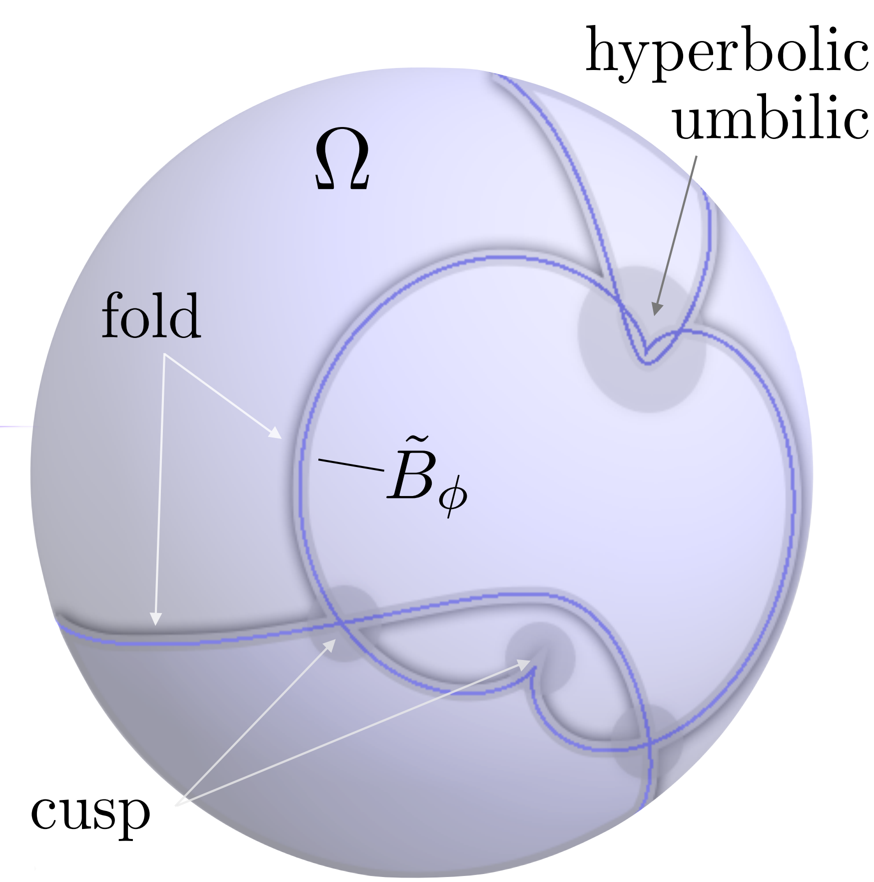
$$\breve{\mathsf{T}}(\boldsymbol{x}^\circ,\beta,\gamma) \;=\; \int_\Omega \mathsf{T}(\boldsymbol{x}^\circ,\beta,\gamma)\,\mathrm{d}\Omega_{\boldsymbol{d}}$$

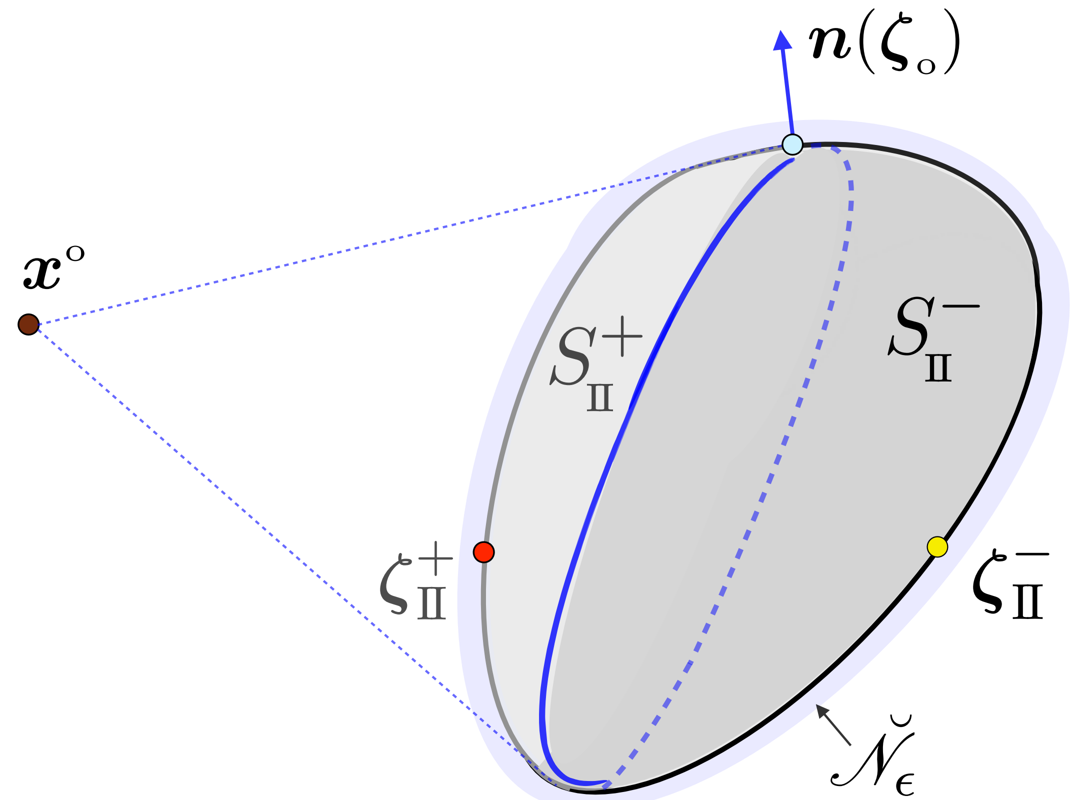
$$|\boldsymbol{d}^*\!\cdot\boldsymbol{n}| = |(\widehat{\boldsymbol{\zeta}-\boldsymbol{x}^\circ})\cdot\boldsymbol{n}(\boldsymbol{\zeta})|$$
$$r = |\boldsymbol{x}^\circ-\boldsymbol{\zeta}|$$

$$\breve{\mathsf{T}} \overset{k^\nu}{=} 1_{\breve{\mathscr{N}}_\epsilon}(\boldsymbol{x}^\circ)\int_\Omega \mathsf{T}^\star\,\mathrm{d}\Omega_{\boldsymbol{d}} + \int_{\tilde{B}_\phi}\mathsf{T}^c\,\mathrm{d}\Omega_{\boldsymbol{d}} + \int_{S^\pm}\mathsf{T}^{\mathrm{II}^\pm}\frac{|\boldsymbol{d}^*\!\cdot\boldsymbol{n}|}{r^2}\,\mathrm{d}S_\zeta$$

non-degenerate SPs

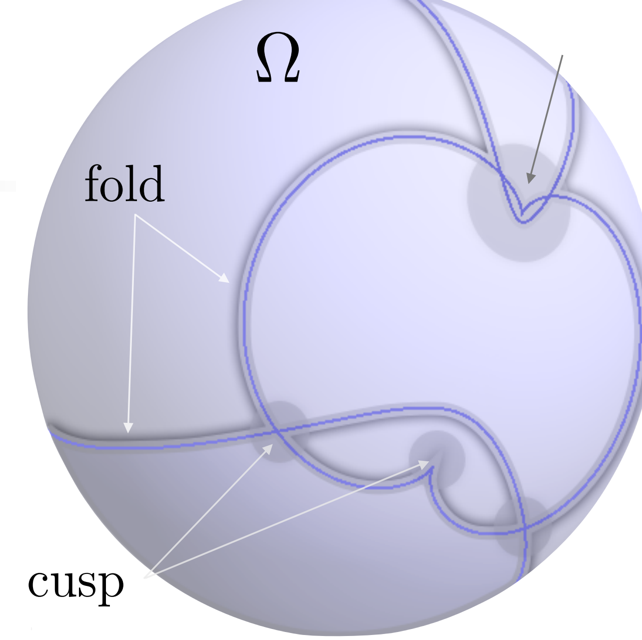
$$1_{\mathscr{G}(\boldsymbol{d})}(\boldsymbol{x}^\circ)\int_\Omega \mathsf{T}^{\mathrm{II}^+}\,\mathrm{d}\Omega_{\boldsymbol{d}} \;=\; -\int_{S^+_{\mathrm{II}}}\mathsf{T}^{\mathrm{II}^+}\frac{|\boldsymbol{d}^*\!\cdot\boldsymbol{n}|}{r^2}\,\mathrm{d}S_\zeta$$





$$\int_{S^\pm}\mathsf{T}^{\mathrm{II}^\pm}\frac{|\boldsymbol{d}^*\!\cdot\boldsymbol{n}|}{r^2}\,\mathrm{d}S_\zeta \;=\; O(k^\alpha), \qquad 0\leqslant\alpha\leqslant\tfrac{1}{3}$$

Full aperture "caustics & $\boldsymbol{x}^\star$



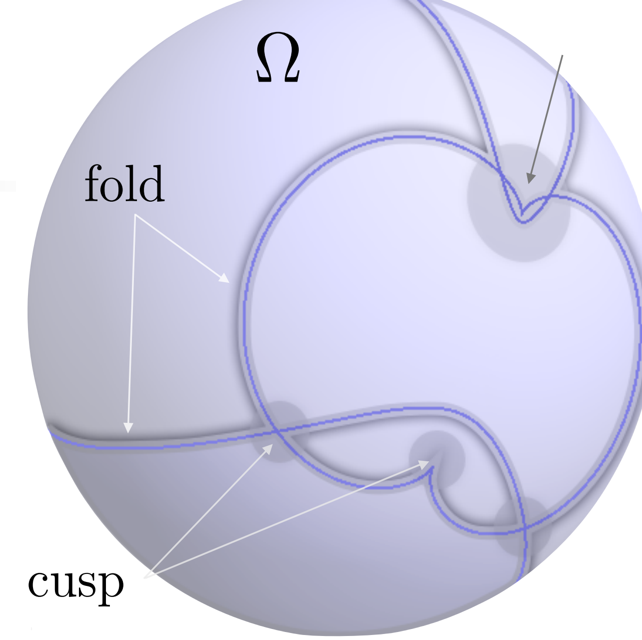
degenerate SPs

$$\int_{\tilde{B}_\phi} \mathsf{T}^c \; d\Omega_{\boldsymbol{d}} \;=\; O(k^\alpha), \qquad \tfrac{1}{4} \leqslant \alpha \leqslant \tfrac{2}{3}$$

Catastrophe	corank	cod	universal unfolding	μ	$\sigma_m^{\min}$	$\mathsf{T}^c(\boldsymbol{x}^\circ,\cdot,\cdot)$
Fold	1	1	$\pm s^2 + t^3/3 + ct$	1/6	2/3	$O(k^{7/6})$
Cusp	1	2	$\pm s^2 + t^4 + c_2t^2 + c_1t$	1/4	1/2	$O(k^{5/4})$
Swallowtail	1	3	$\pm s^2 + t^5 + c_3t^3 + c_2t^2 + c_1t$	3/10	2/5	$O(k^{13/10})$
Hyp. umbilic	2	3	$s^3 + t^3 + c_3st + c_2t + c_1s$	1/3	1/3	$O(k^{4/3})$
Ell. umbilic	2	3	$s^3 - st^2 + c_3(s^2 + t^2) + c_2t + c_1s$	1/3	1/3	$O(k^{4/3})$

Full aperture

"caustics & $x^\star$

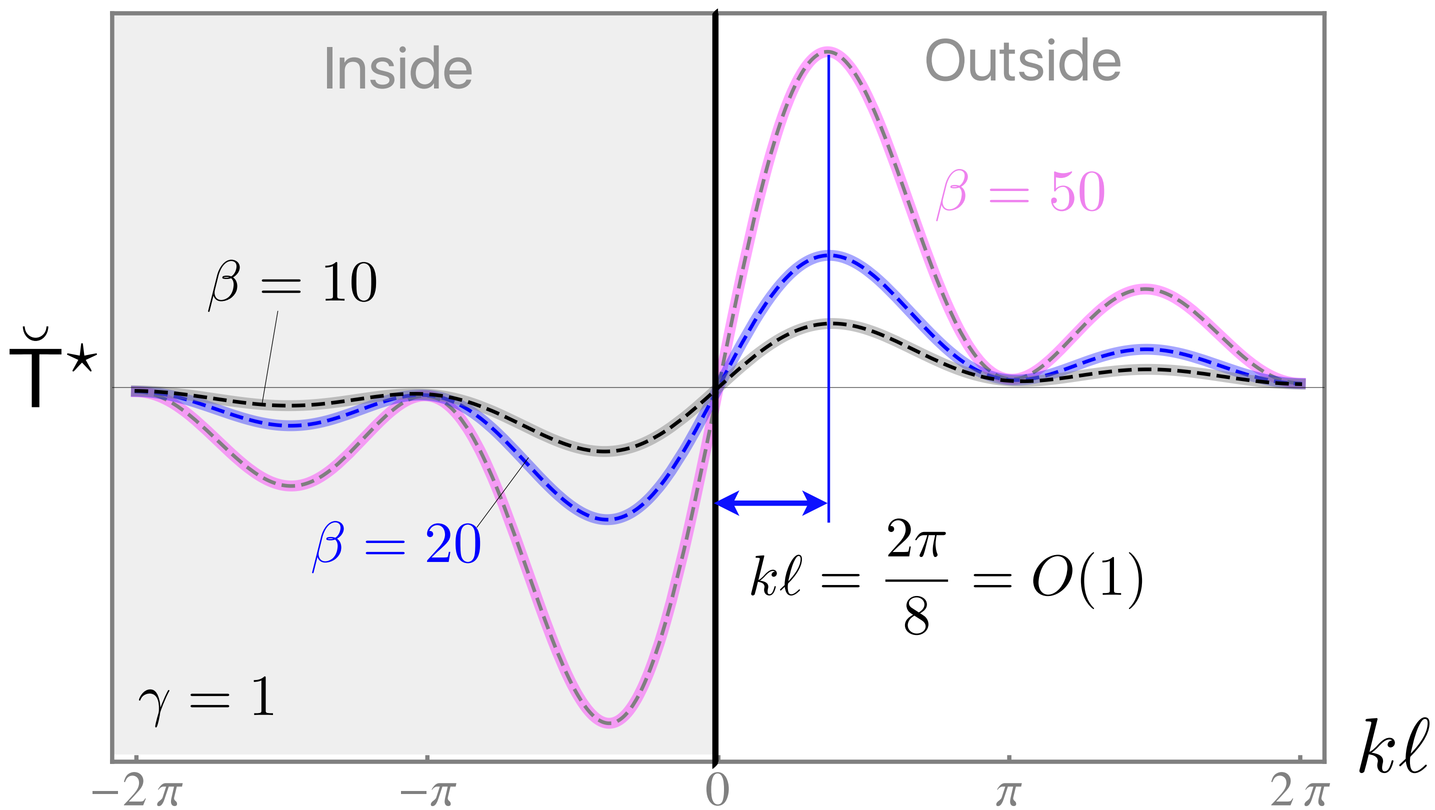


degenerate SPs

$$\int_{\tilde{B}_\phi} \mathsf{T}^c \; d\Omega_{\boldsymbol{d}} \;=\; O(k^\alpha), \qquad \tfrac{1}{4} \leqslant \alpha \leqslant \tfrac{2}{3}$$

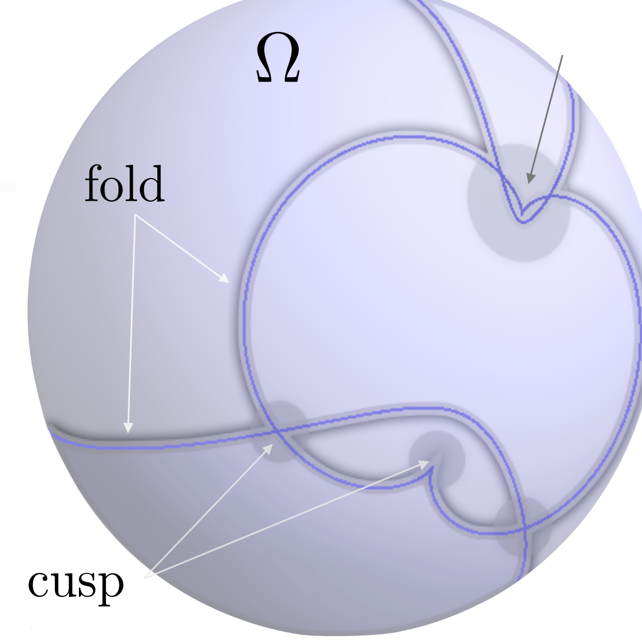
"nearby" SPs

$$\breve{\mathsf{T}}^\star = \int_\Omega \mathsf{T}^\star \, \mathrm{d}\Omega_{\boldsymbol{d}} \stackrel{1}{=} \frac{\pi k}{(k\ell)^3} \left\{ \frac{3(1-\beta)}{2+\beta} \big(k\ell\cos(k\ell) - \sin(k\ell)\big)^2 - (1-\beta\gamma^2)\,(k\ell)^2 \sin(k\ell)^2 \right\}$$



Full aperture

"caustics & $x^\star$"

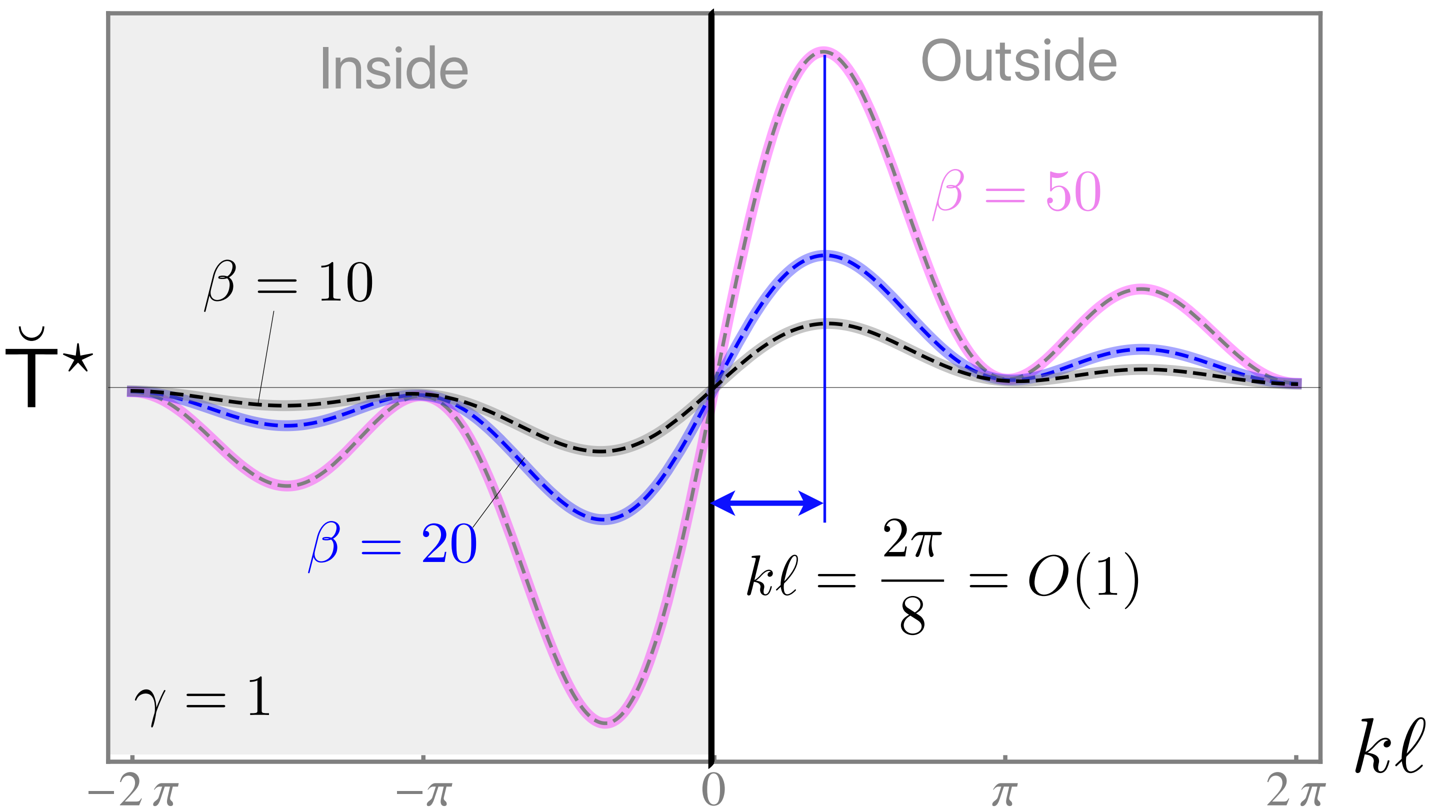


degenerate SPs

$$\int_{\tilde{B}_\phi} \mathsf{T}^c \; d\Omega_{\boldsymbol{d}} \;=\; O(k^\alpha), \qquad \tfrac{1}{4} \leqslant \alpha \leqslant \tfrac{2}{3}$$

"nearby" SPs

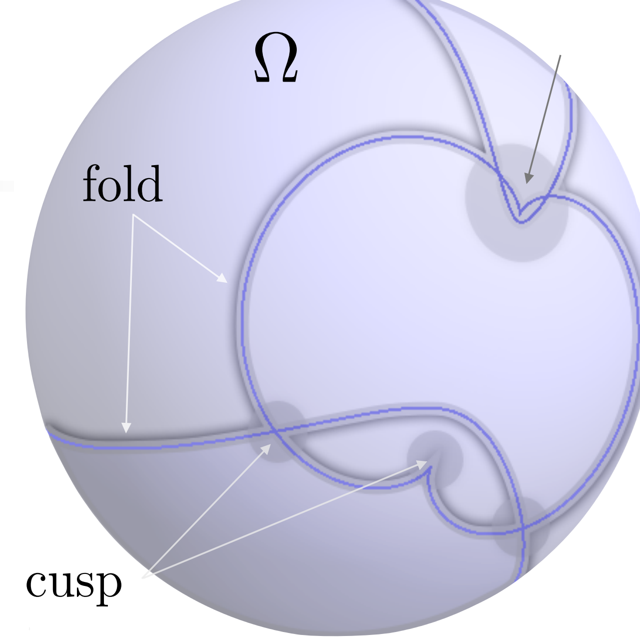
$$\breve{\mathsf{T}}^\star = \int_\Omega \mathsf{T}^\star \, \mathrm{d}\Omega_{\boldsymbol{d}} \overset{1}{=} \frac{\pi k}{(k\ell)^3} \left\{ \frac{3(1-\beta)}{2+\beta} \big(k\ell\cos(k\ell) - \sin(k\ell)\big)^2 - (1-\beta\gamma^2)\,(k\ell)^2 \sin(k\ell)^2 \right\}$$



non-degenerate SPs	$O(k^{\frac{1}{3}})$	
degenerate SPs	$O(k^{\frac{2}{3}})$	45
"nearby" SPs	$O(k)$	300

Full aperture

"caustics & $\boldsymbol{x}^\star$

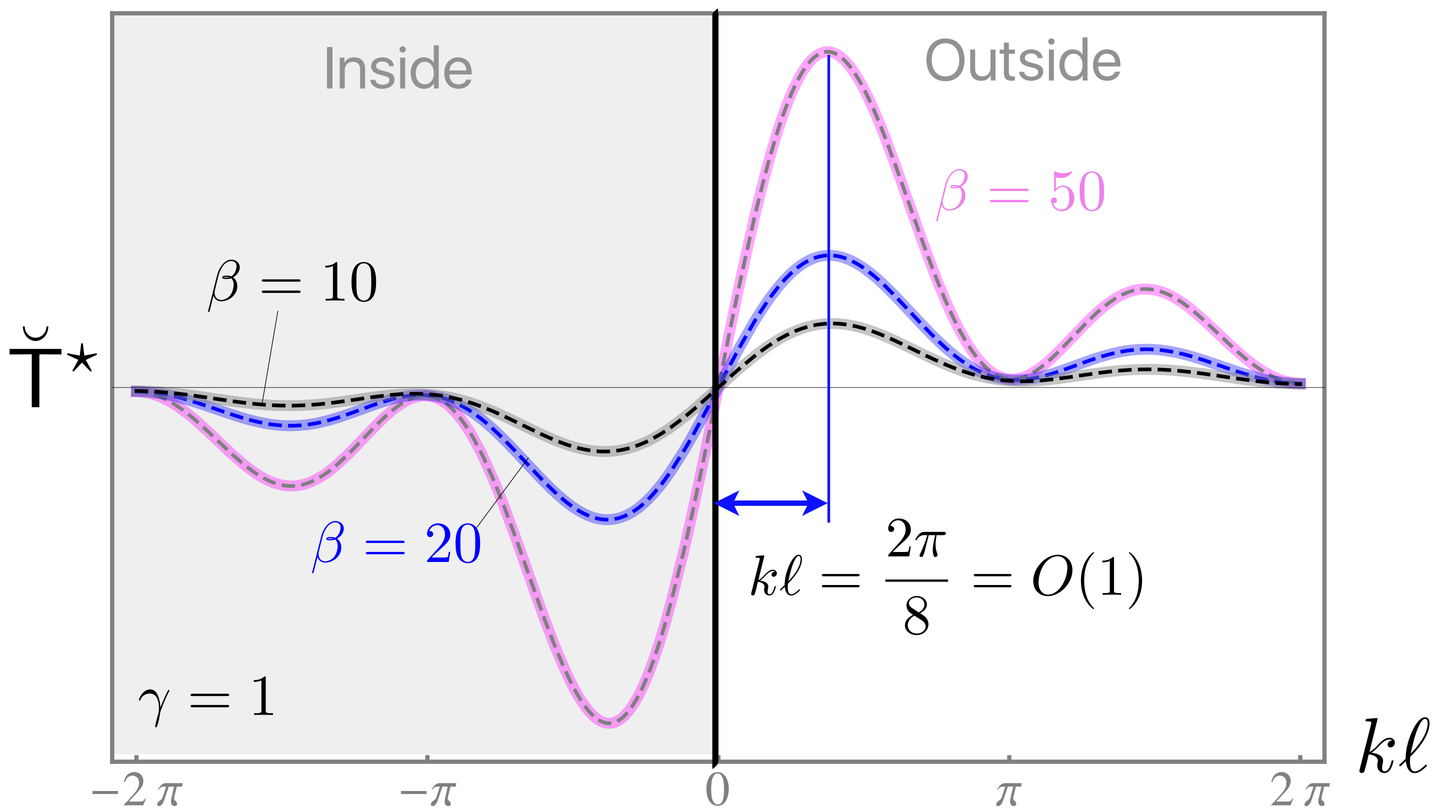


degenerate SPs

$$\int_{\tilde{B}_\phi} \mathsf{T}^c \; d\Omega_{\boldsymbol{d}} \;=\; O(k^\alpha), \qquad \tfrac{1}{4} \leqslant \alpha \leqslant \tfrac{2}{3}$$

"nearby" SPs

$$\breve{\mathsf{T}}^\star = \int_\Omega \mathsf{T}^\star \, \mathrm{d}\Omega_{\boldsymbol{d}} \stackrel{1}{=} \frac{\pi k}{(k\ell)^3} \left\{ \frac{3(1-\beta)}{2+\beta} \big(k\ell \cos(k\ell) - \sin(k\ell)\big)^2 - (1-\beta\gamma^2)\,(k\ell)^2 \sin(k\ell)^2 \right\}$$

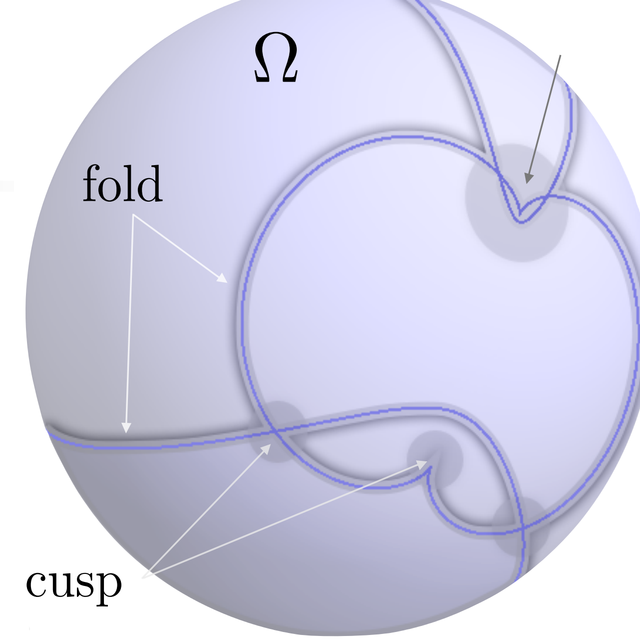


non-degenerate SPs	$O(k^{\frac{1}{3}})$	
degenerate SPs	$O(k^{\frac{2}{3}})$	45
"nearby" SPs	$O(k)$	300

$$\breve{\mathsf{T}}(\boldsymbol{x}^\circ, \beta, \gamma) \stackrel{k^\alpha}{=} 1_{\breve{\mathcal{N}}_\epsilon}(\boldsymbol{x}^\circ)\, \breve{\mathsf{T}}^\star(k\ell, \beta, \gamma)$$

Full aperture

"caustics & $x^\star$

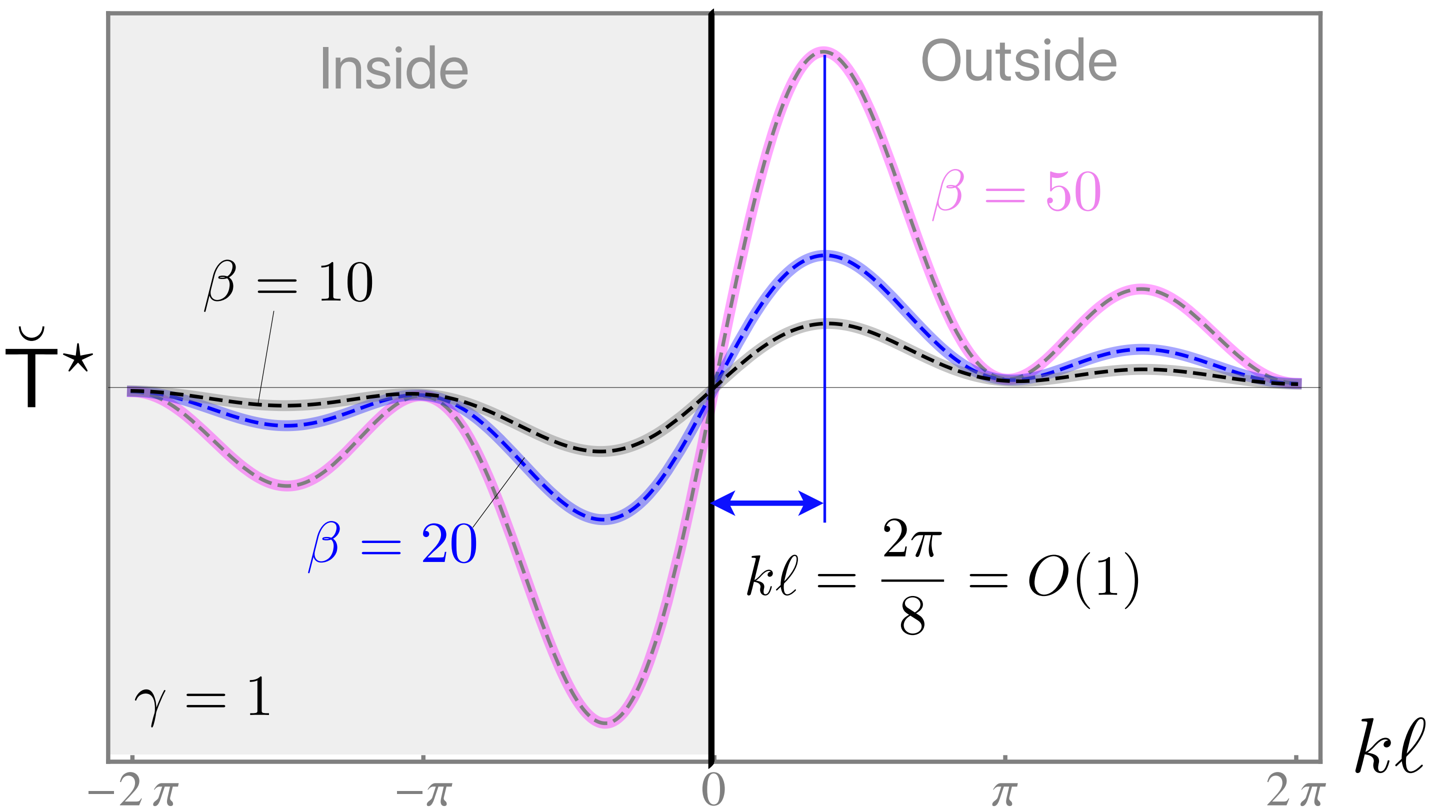


degenerate SPs

$$\int_{\tilde{B}_\phi} \mathsf{T}^c \; d\Omega_{\boldsymbol{d}} \;=\; O(k^\alpha), \qquad \tfrac{1}{4} \leqslant \alpha \leqslant \tfrac{2}{3}$$

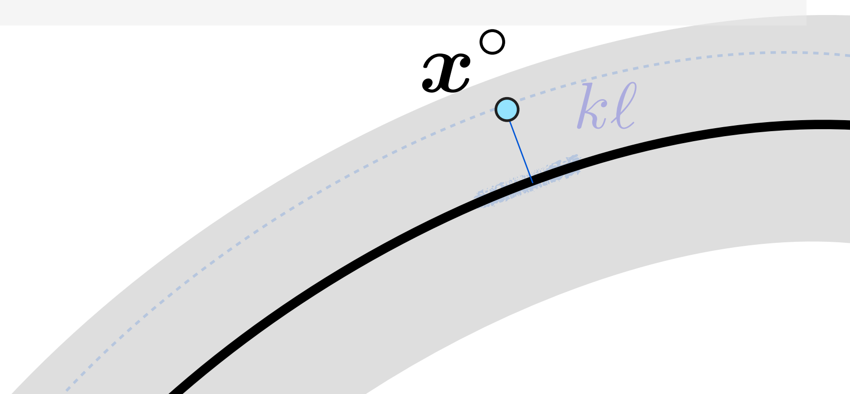
"nearby" SPs

$$\breve{\mathsf{T}}^\star = \int_\Omega \mathsf{T}^\star \, d\Omega_{\boldsymbol{d}} \stackrel{1}{=} \frac{\pi k}{(k\ell)^3} \left\{ \frac{3(1-\beta)}{2+\beta} \big(k\ell\cos(k\ell) - \sin(k\ell)\big)^2 - (1-\beta\gamma^2)\,(k\ell)^2 \sin(k\ell)^2 \right\}$$



non-degenerate SPs	$O(k^{\frac{1}{3}})$	
degenerate SPs	$O(k^{\frac{2}{3}})$	45
"nearby" SPs	$O(k)$	300

✓ The "nearby" SP contribution tracks the shape of the boundary

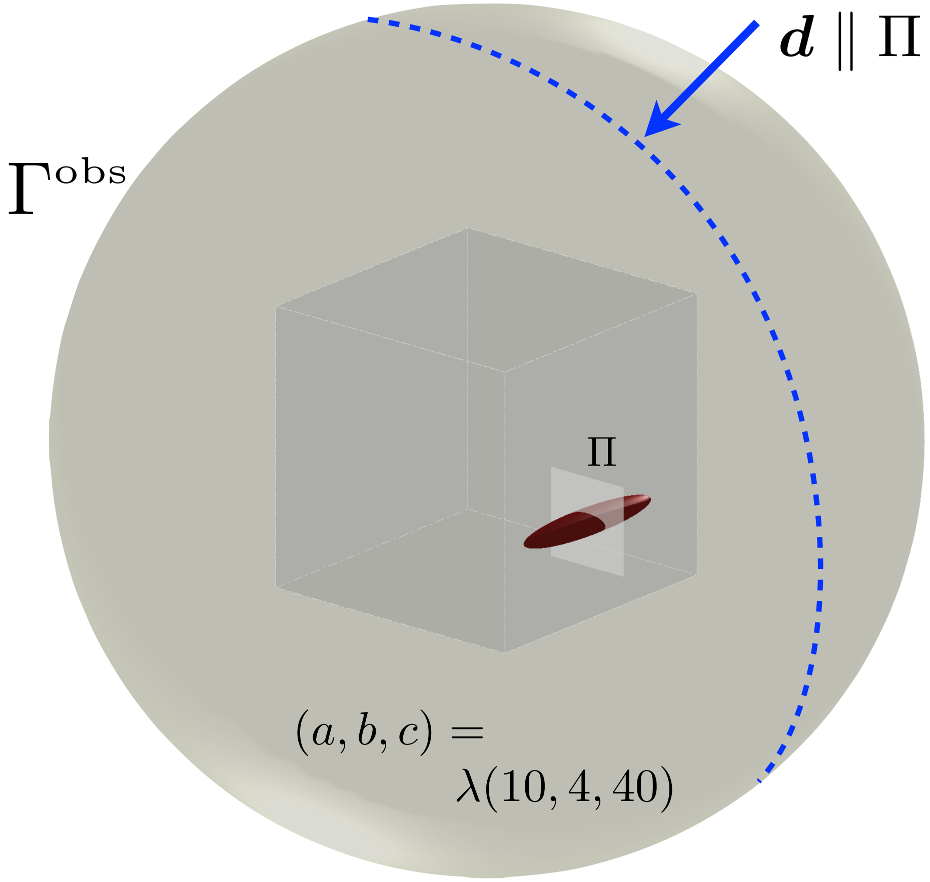


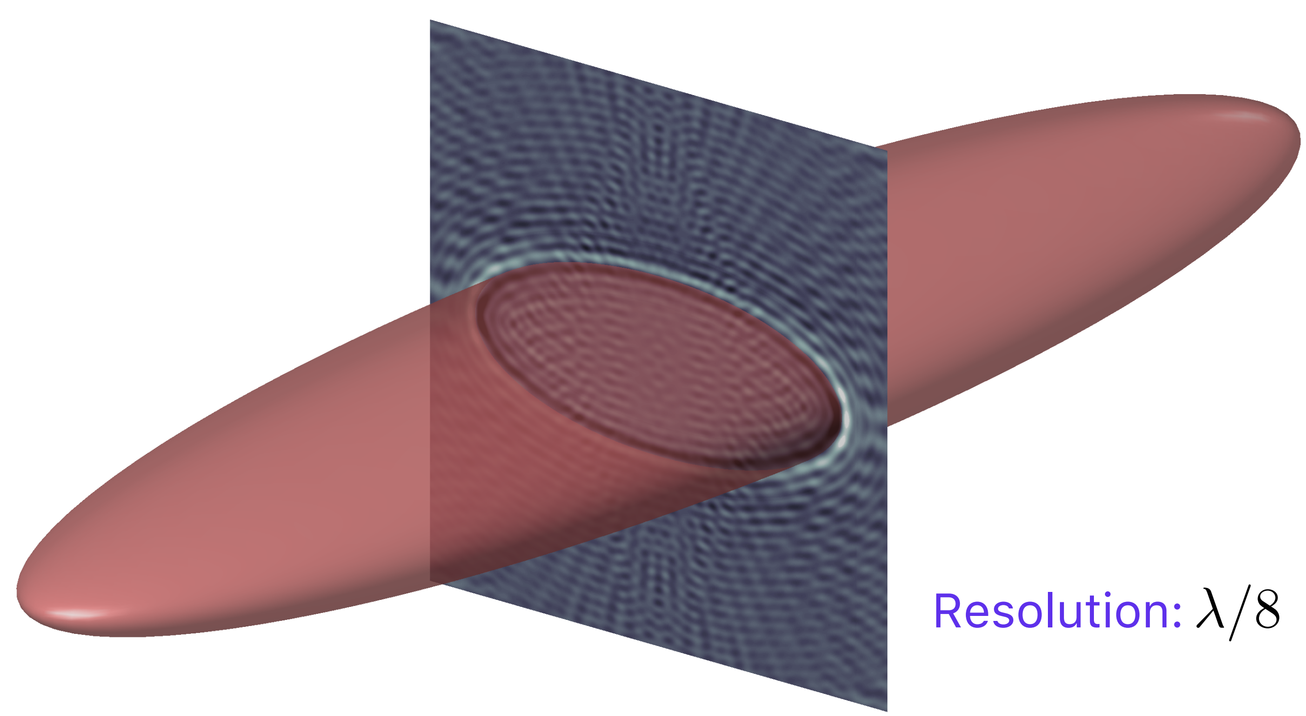
$$\breve{\mathsf{T}}(\boldsymbol{x}^\circ, \beta, \gamma) \stackrel{k^\alpha}{=} 1_{\breve{\mathcal{N}}_\epsilon}(\boldsymbol{x}^\circ)\, \breve{\mathsf{T}}^\star(k\ell, \beta, \gamma)$$

TS: $\beta = 50, \gamma = 1$

Full aperture

Dirichlet obstacle



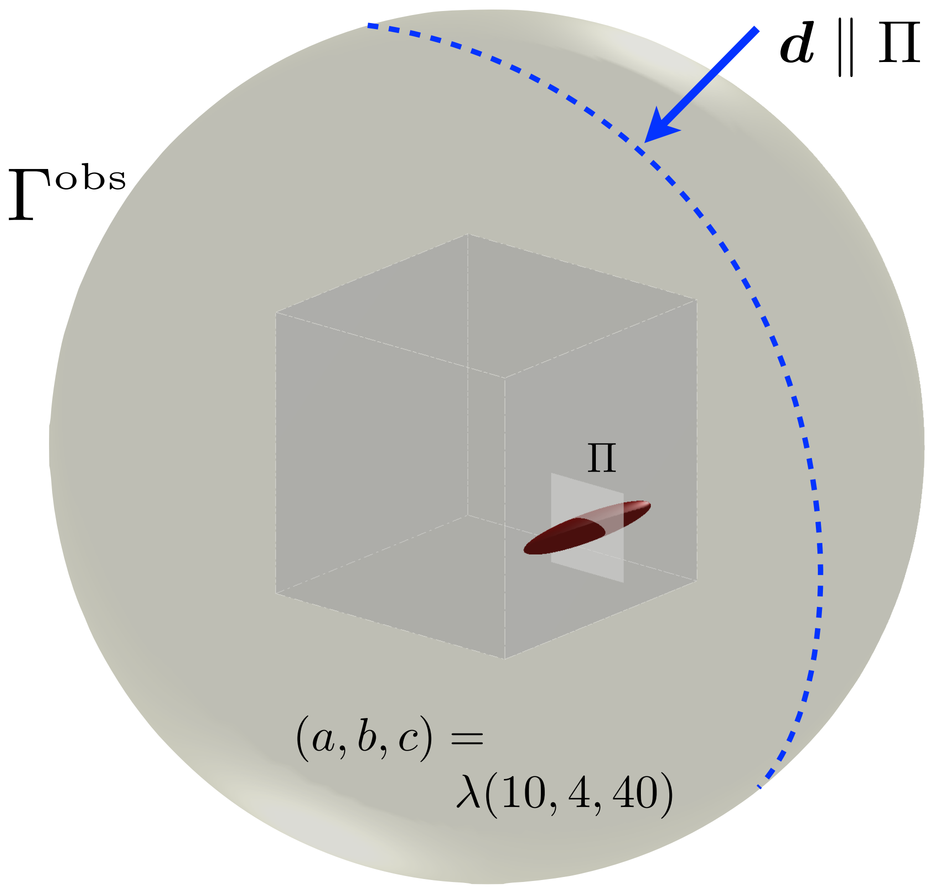


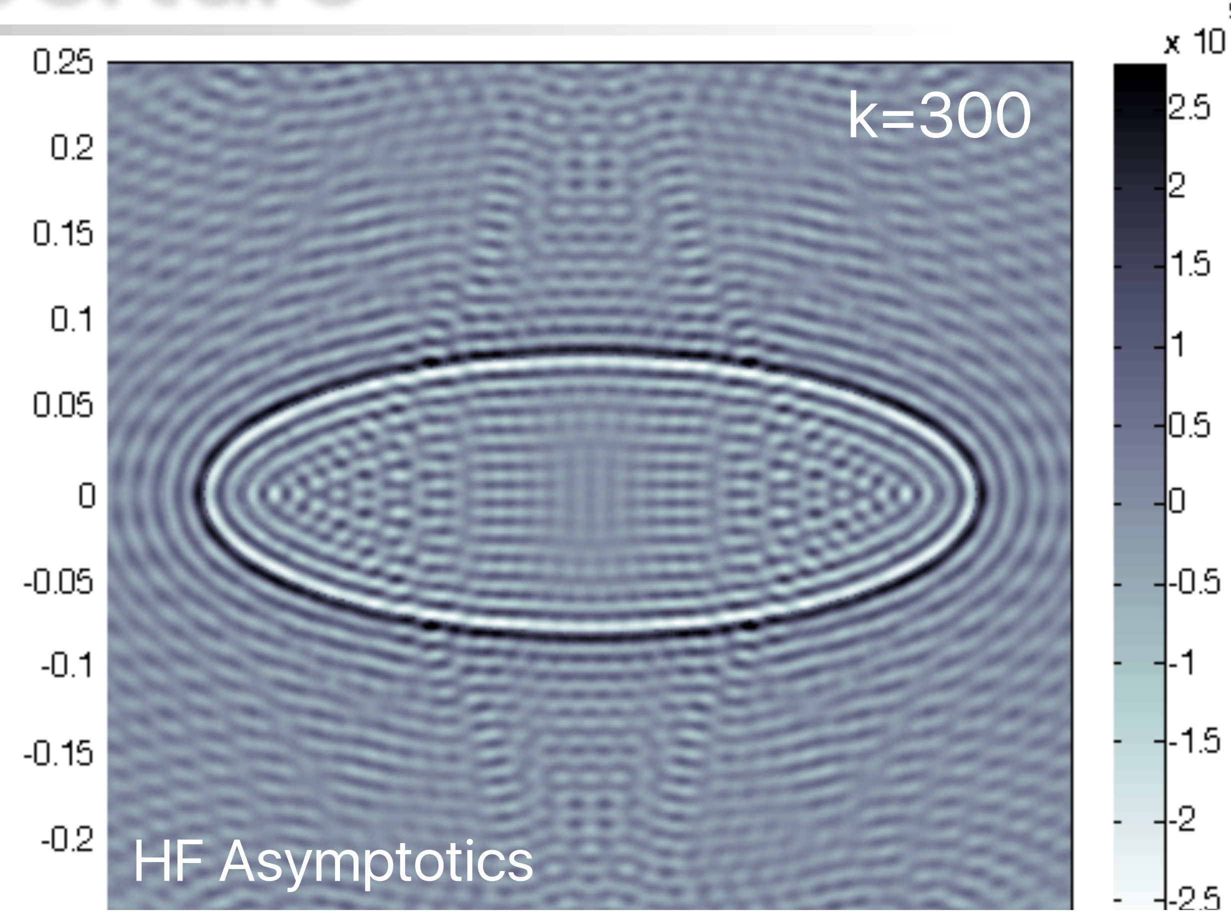
Resolution: $\lambda/8$

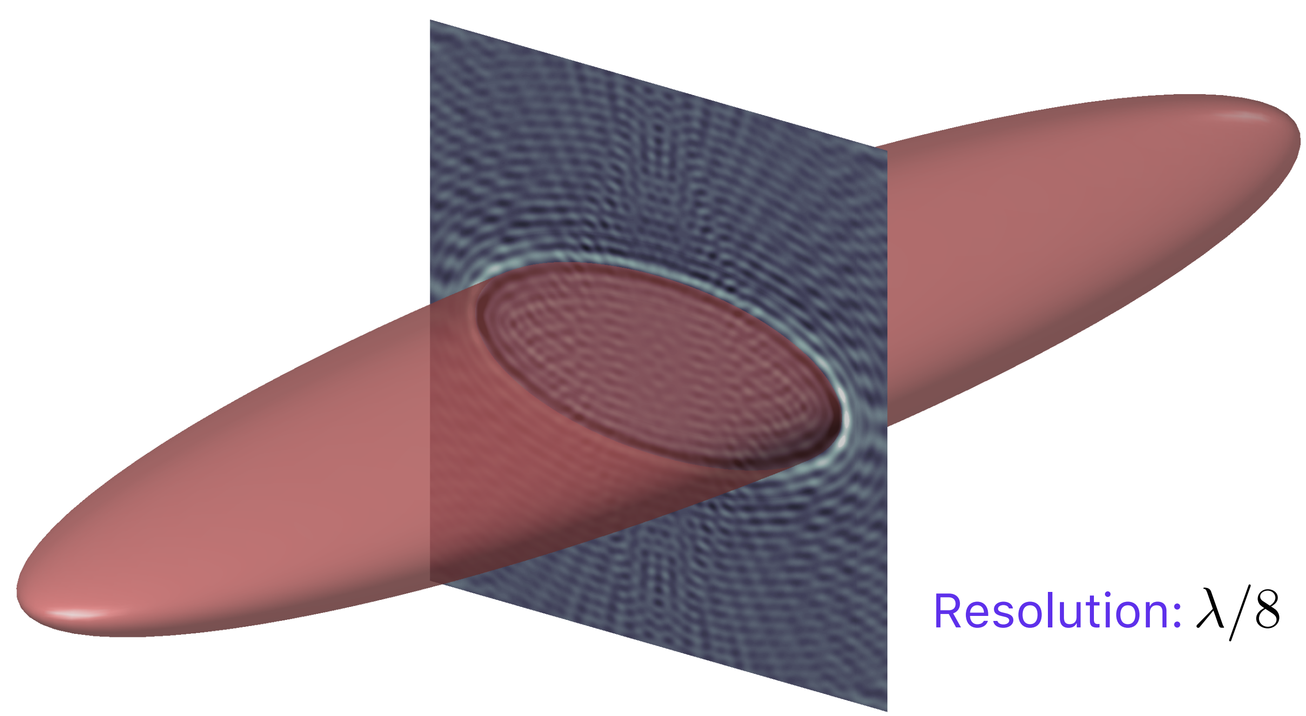
TS: $\beta = 50, \gamma = 1$

Full aperture

Dirichlet obstacle





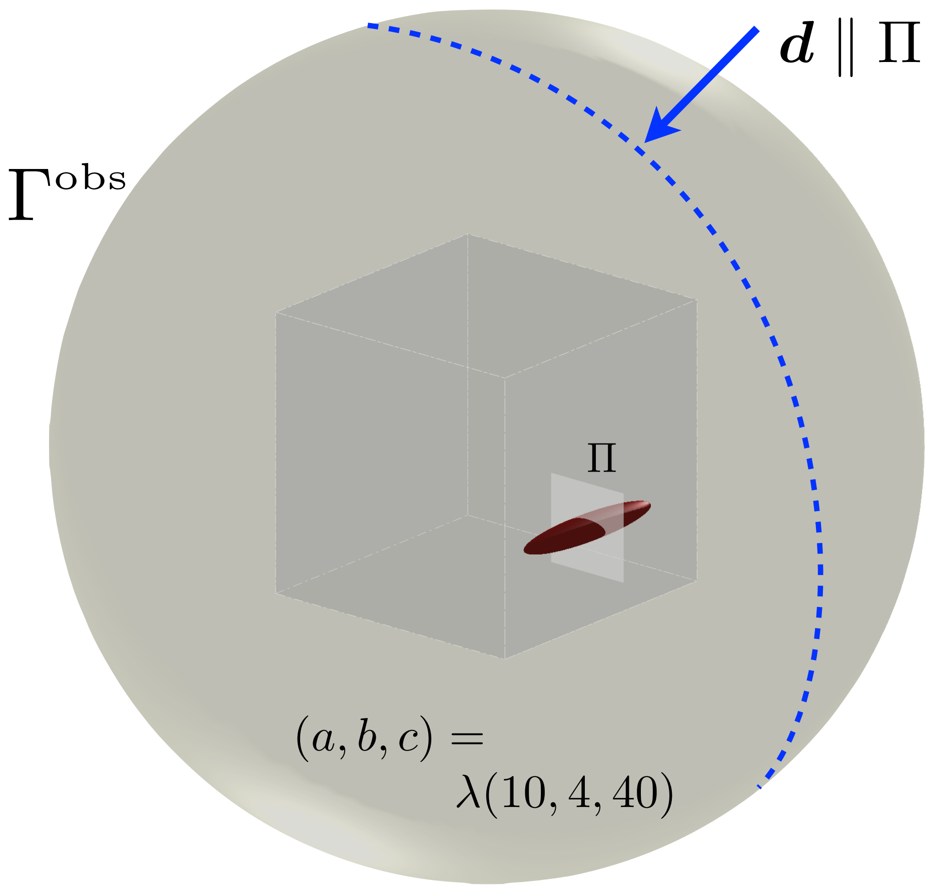


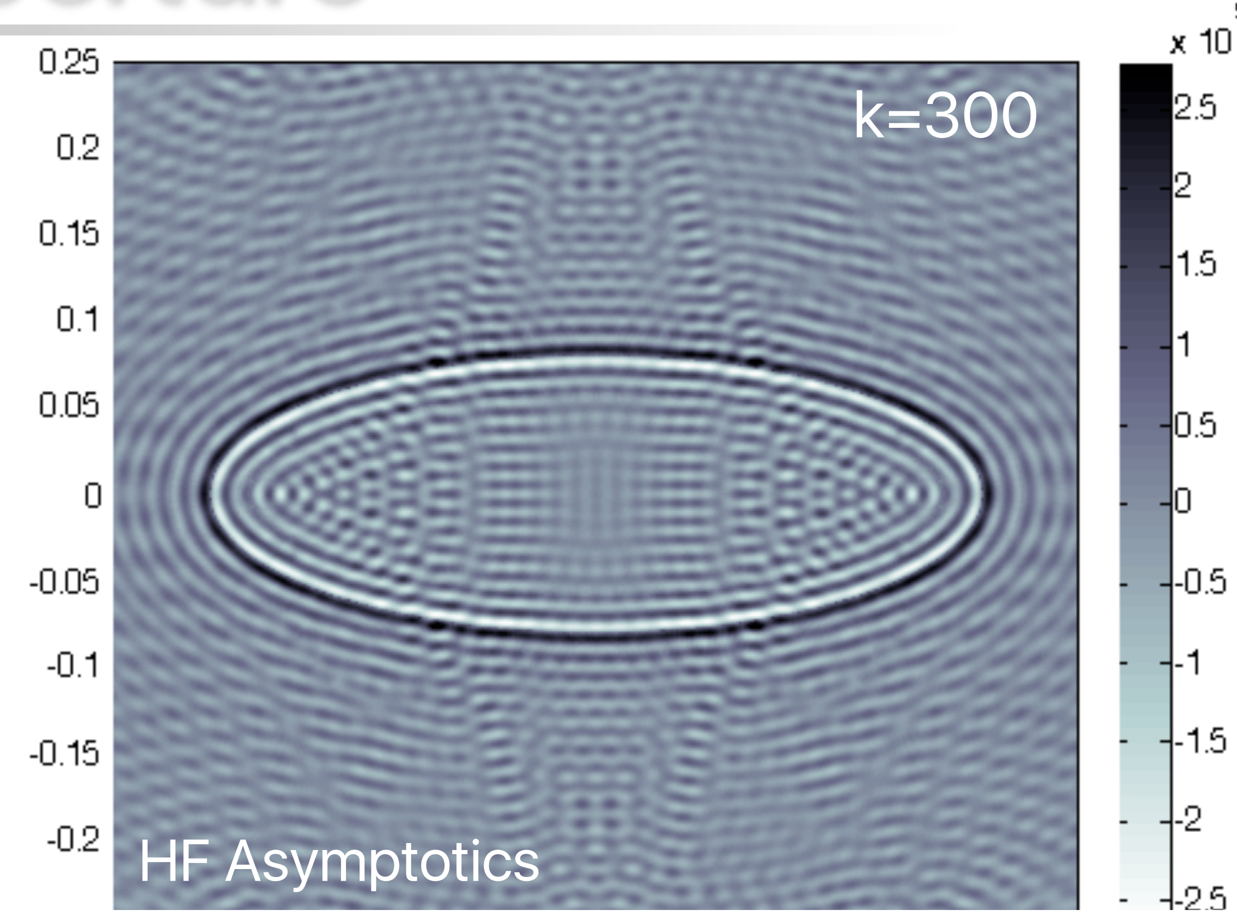
Resolution: $\lambda/8$

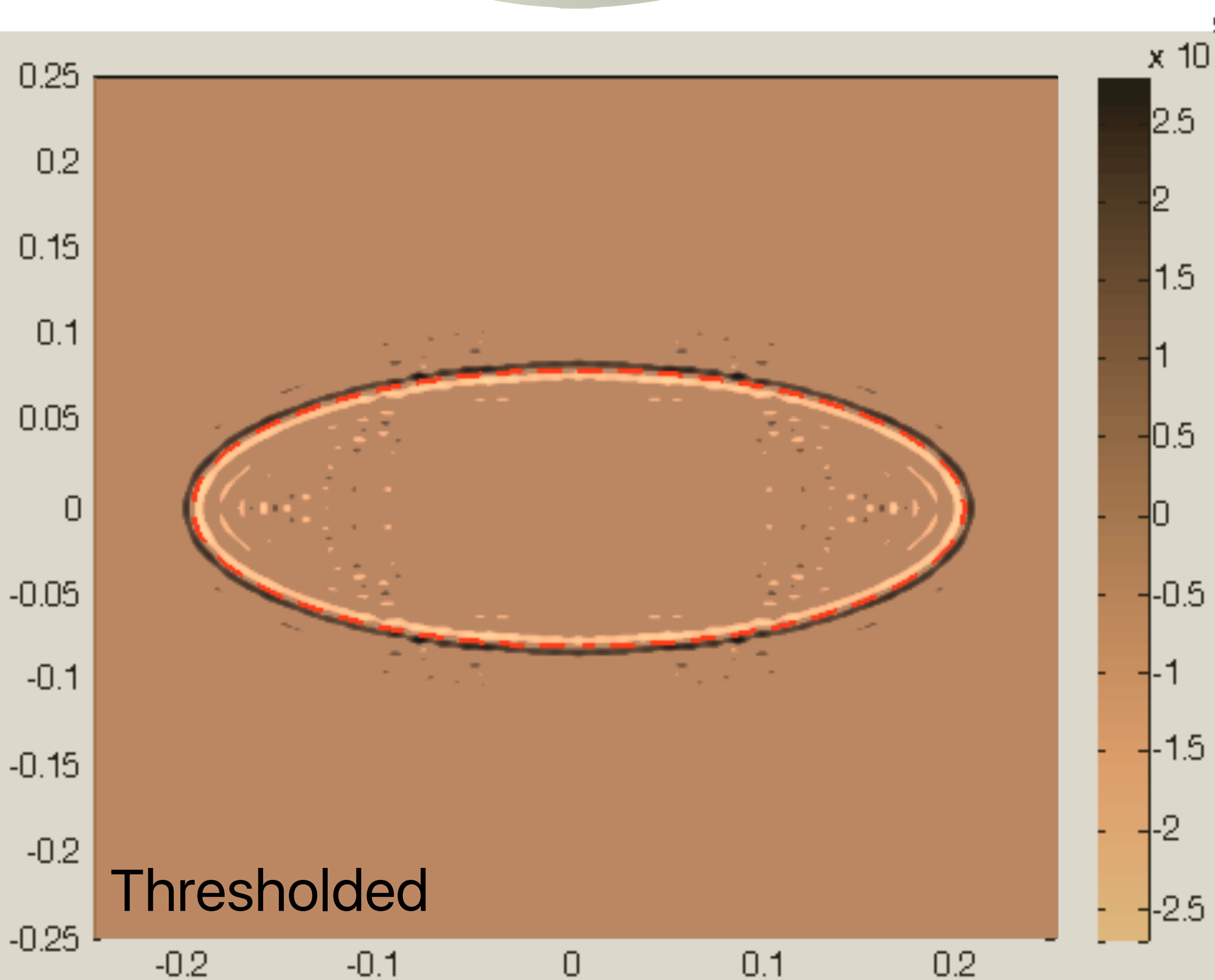
TS: $\beta = 50, \gamma = 1$

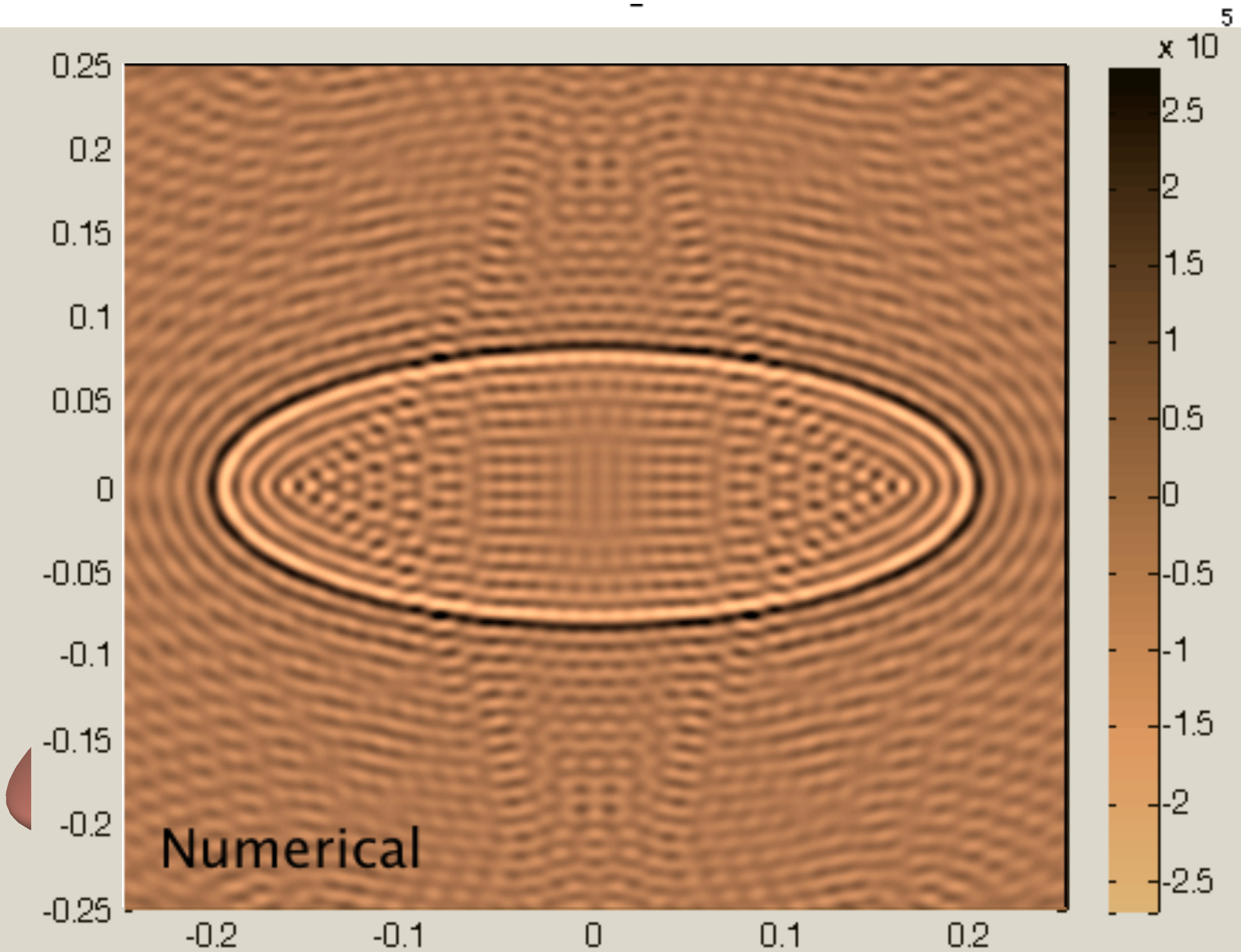
Full aperture

Dirichlet obstacle





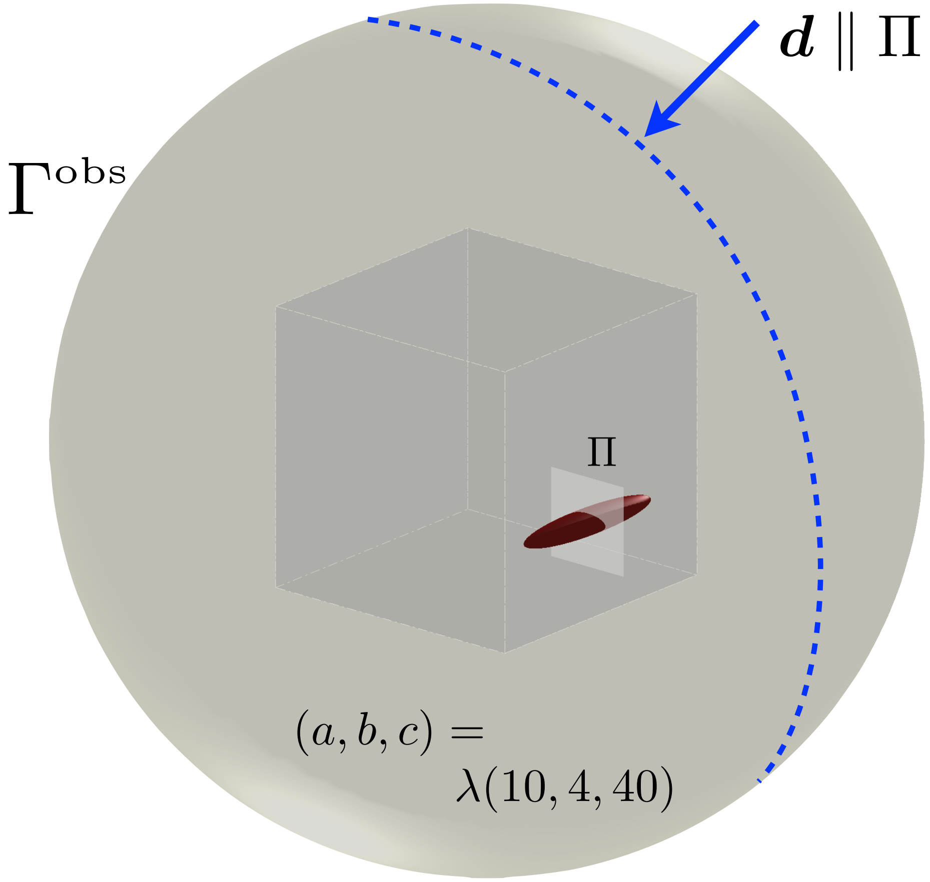


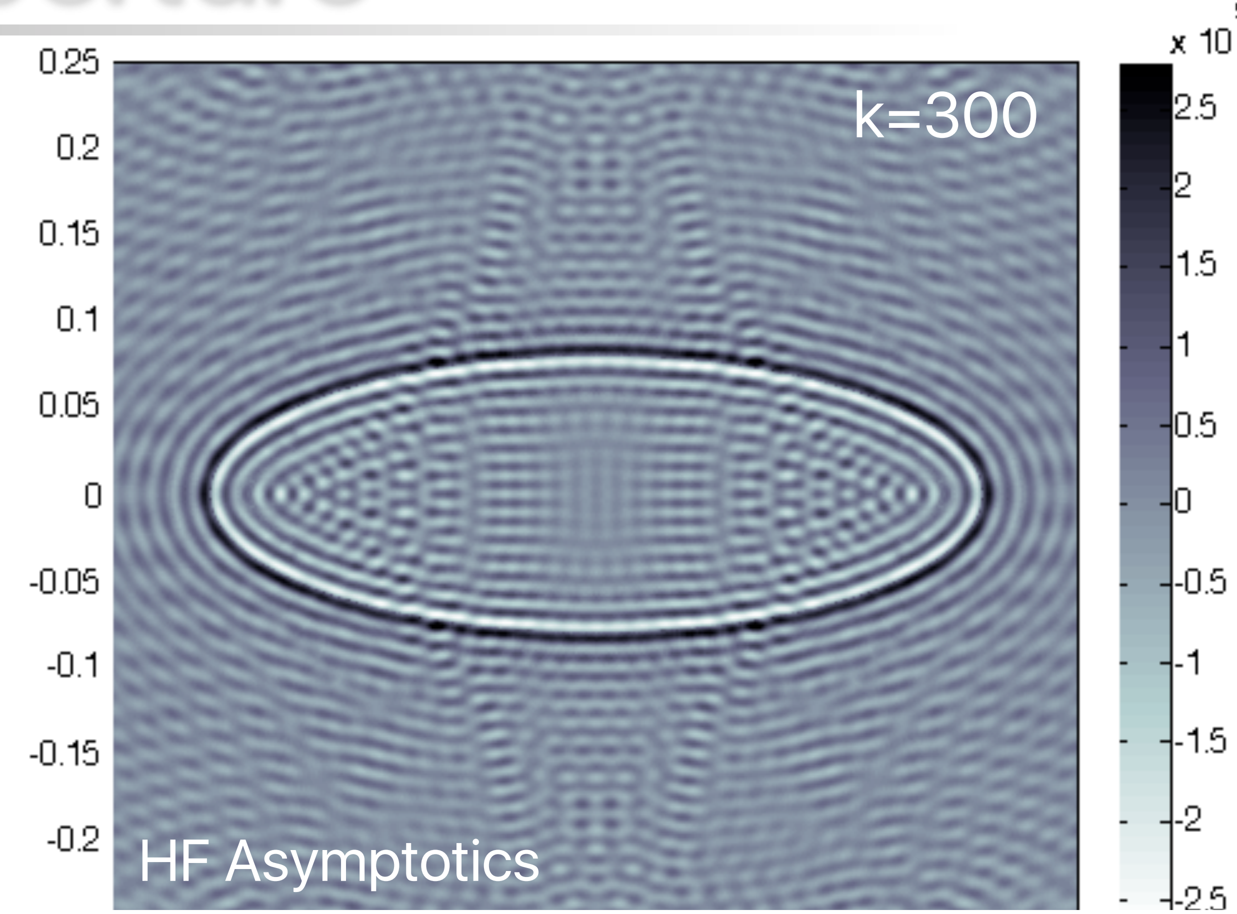


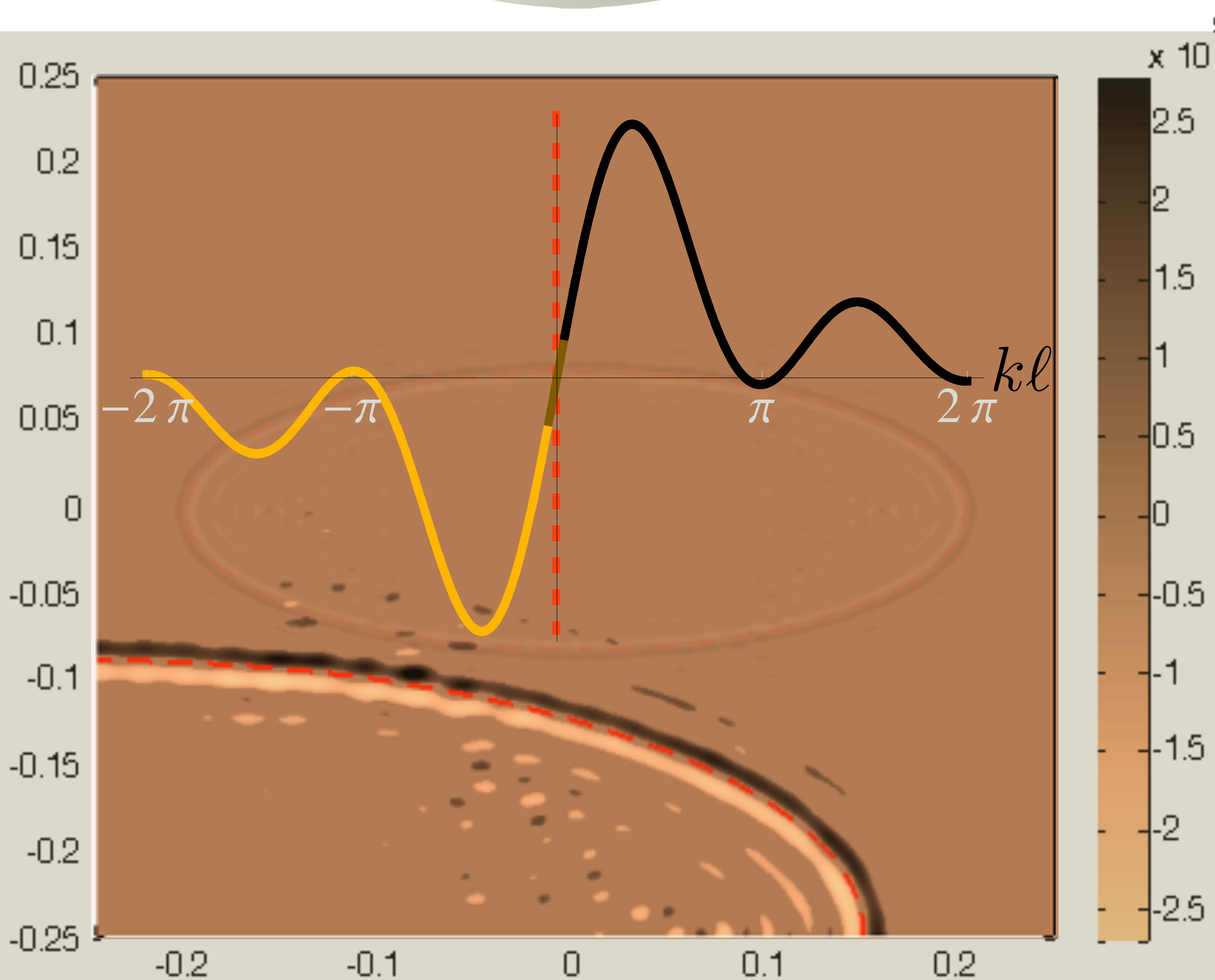
TS: $\beta = 50, \gamma = 1$

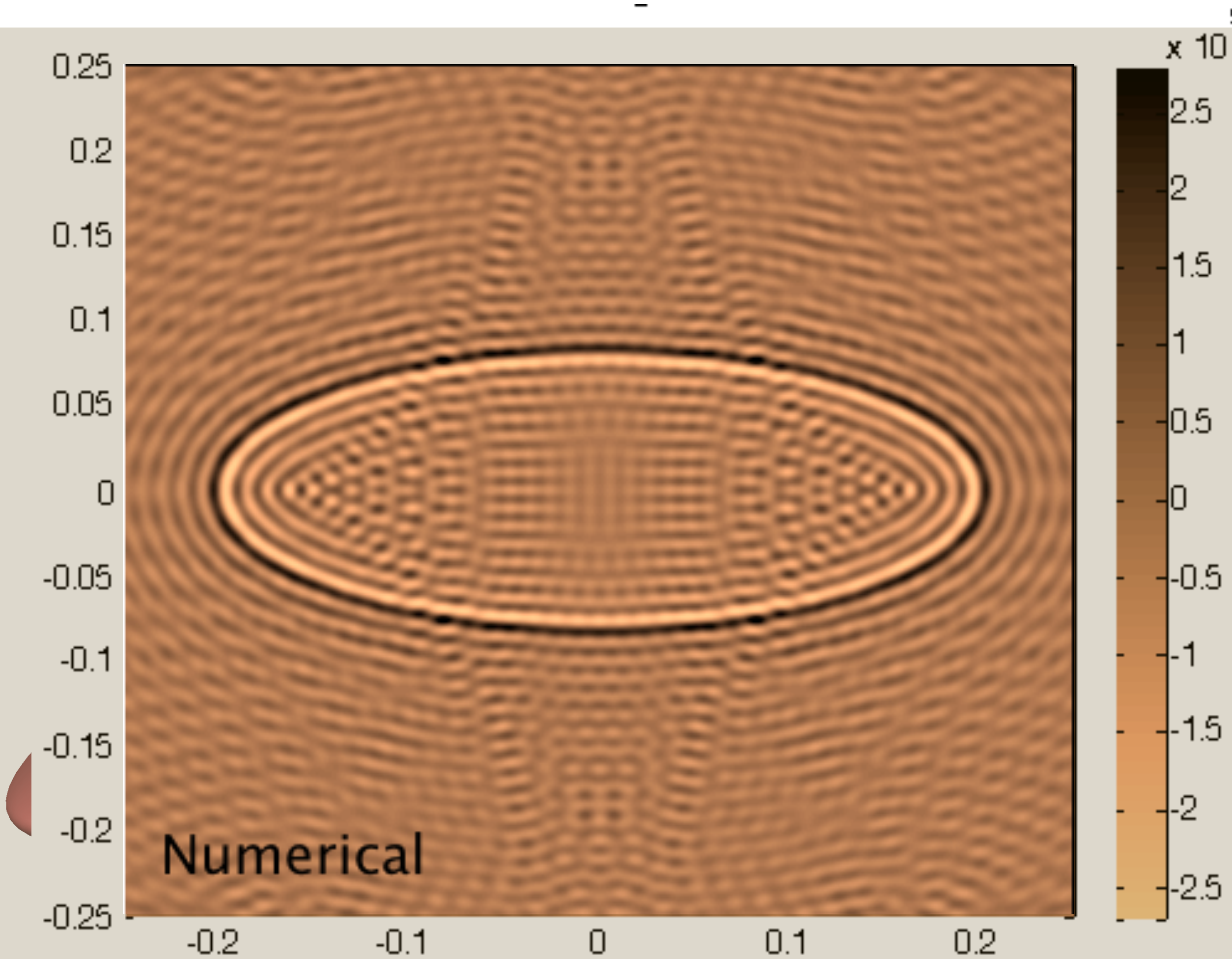
Full aperture

Dirichlet obstacle

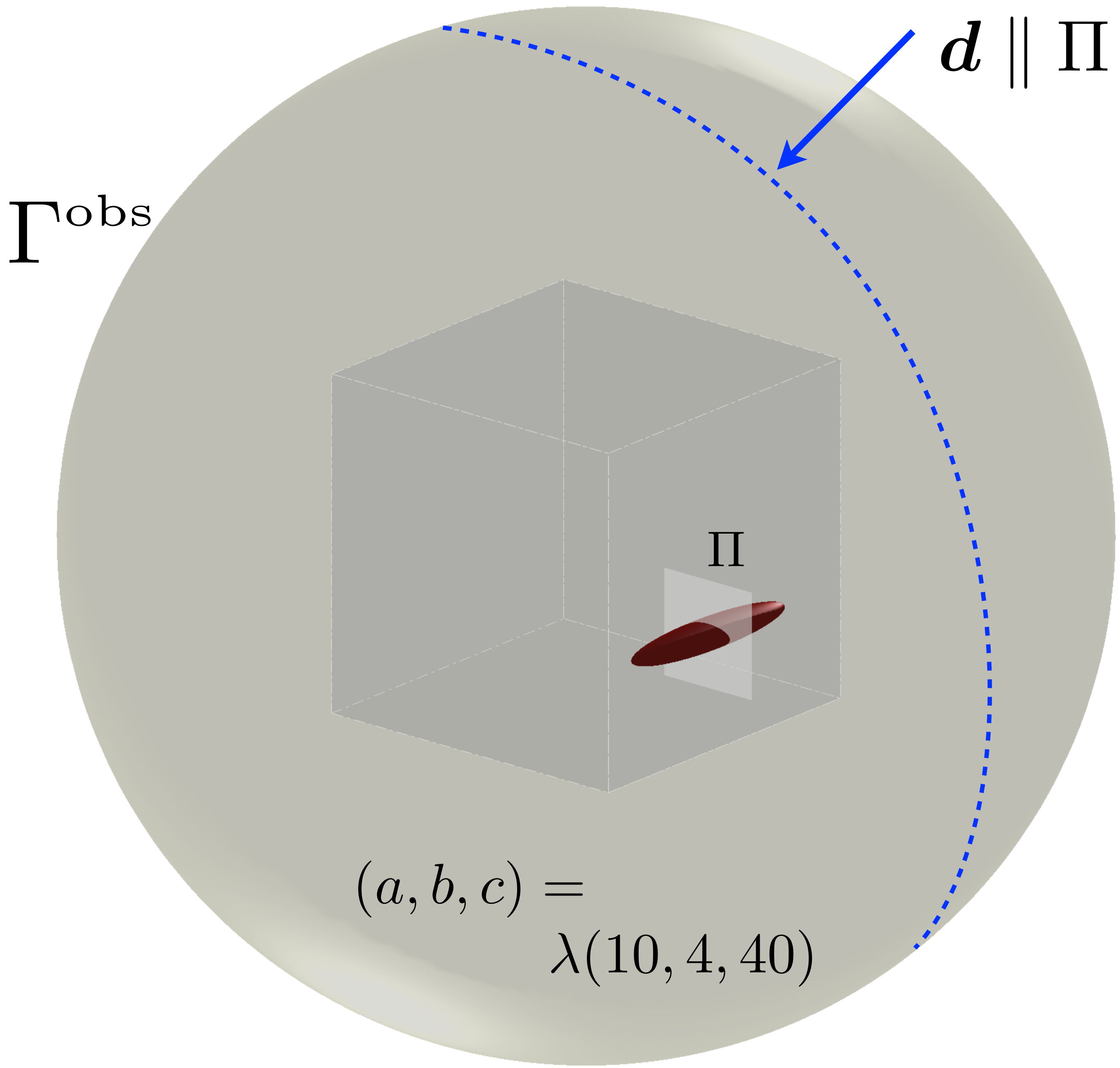




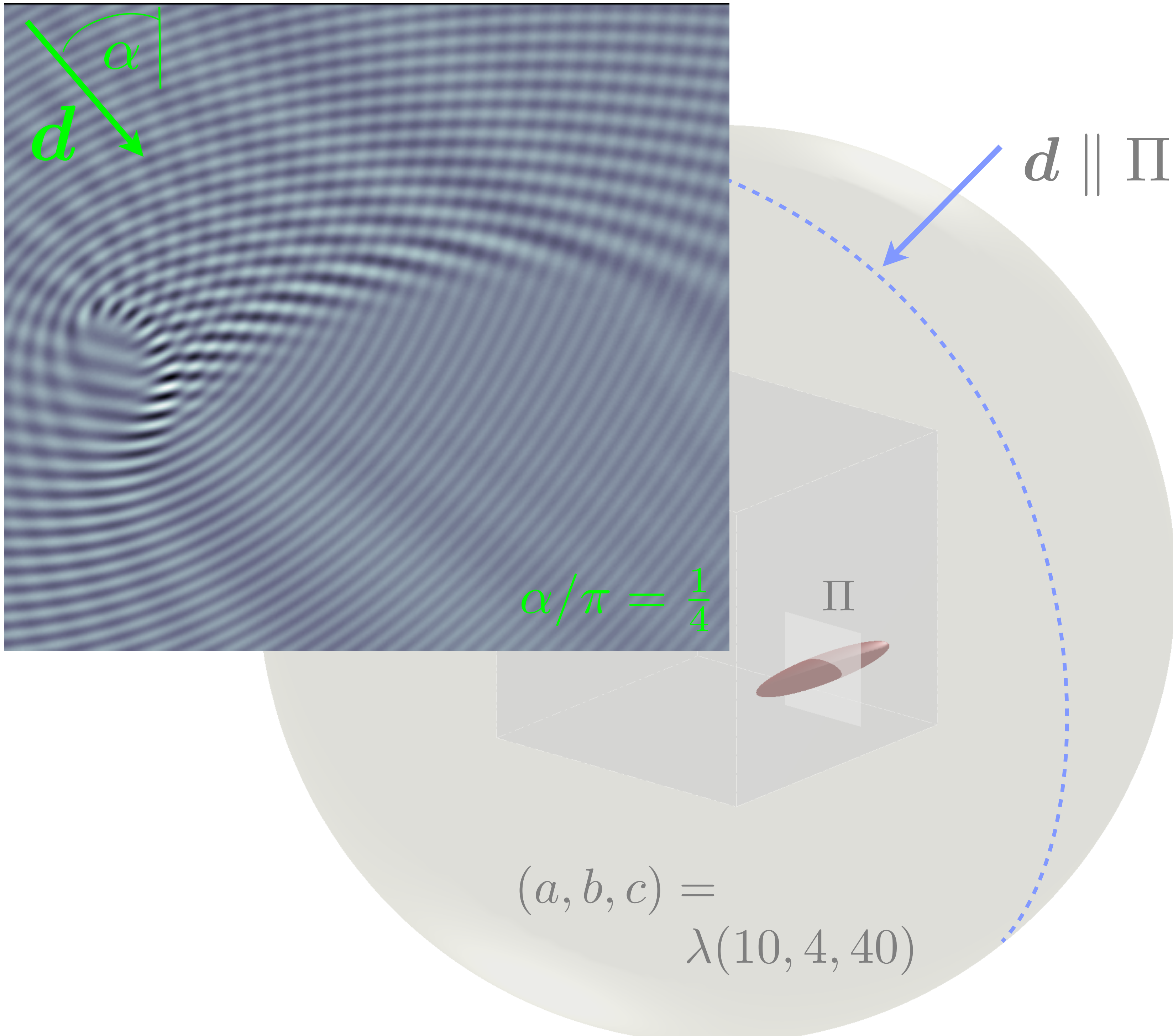


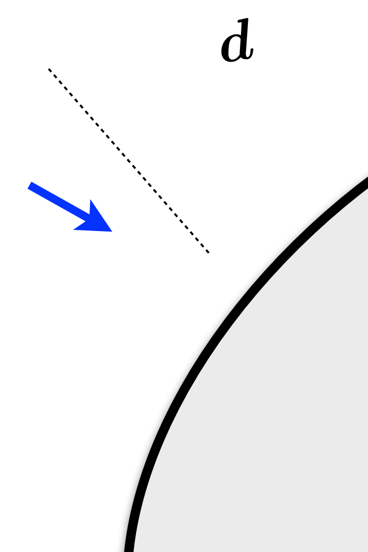


Constructive Interference

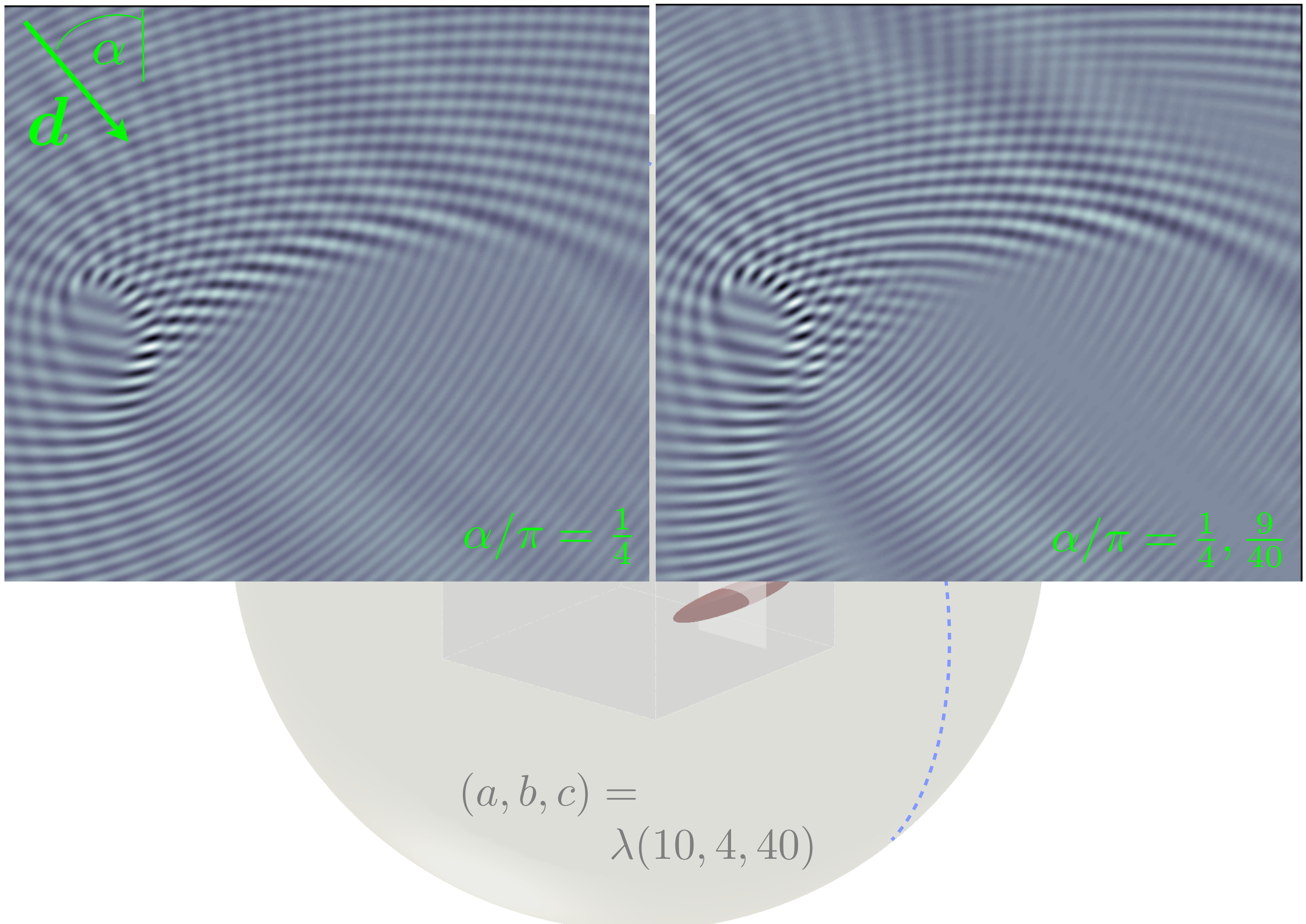


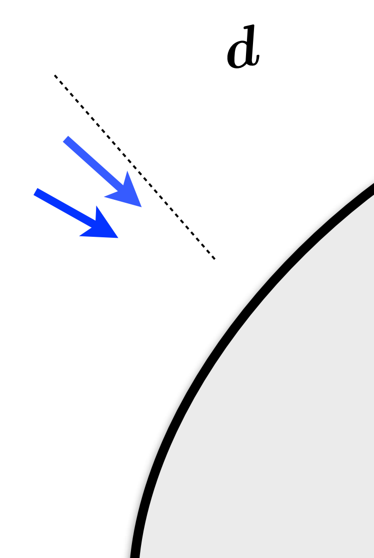
Constructive Interference



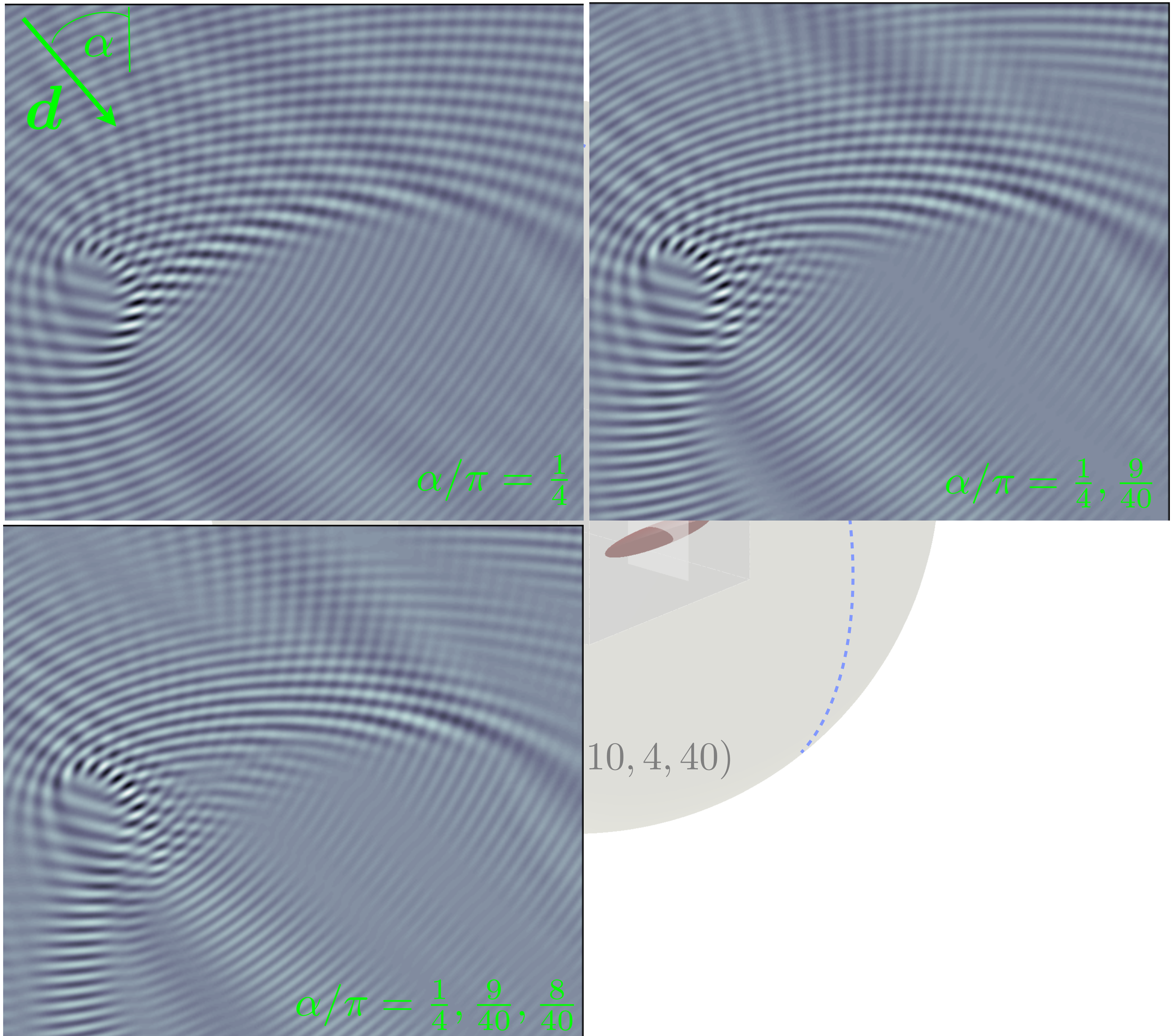


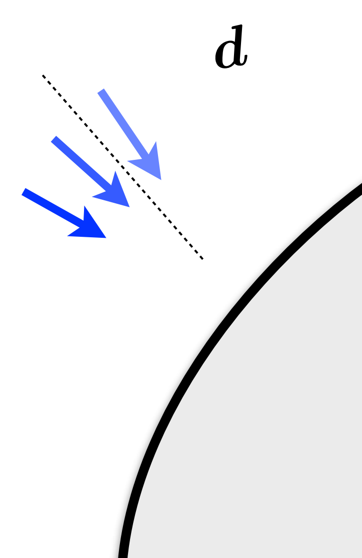
Constructive Interference



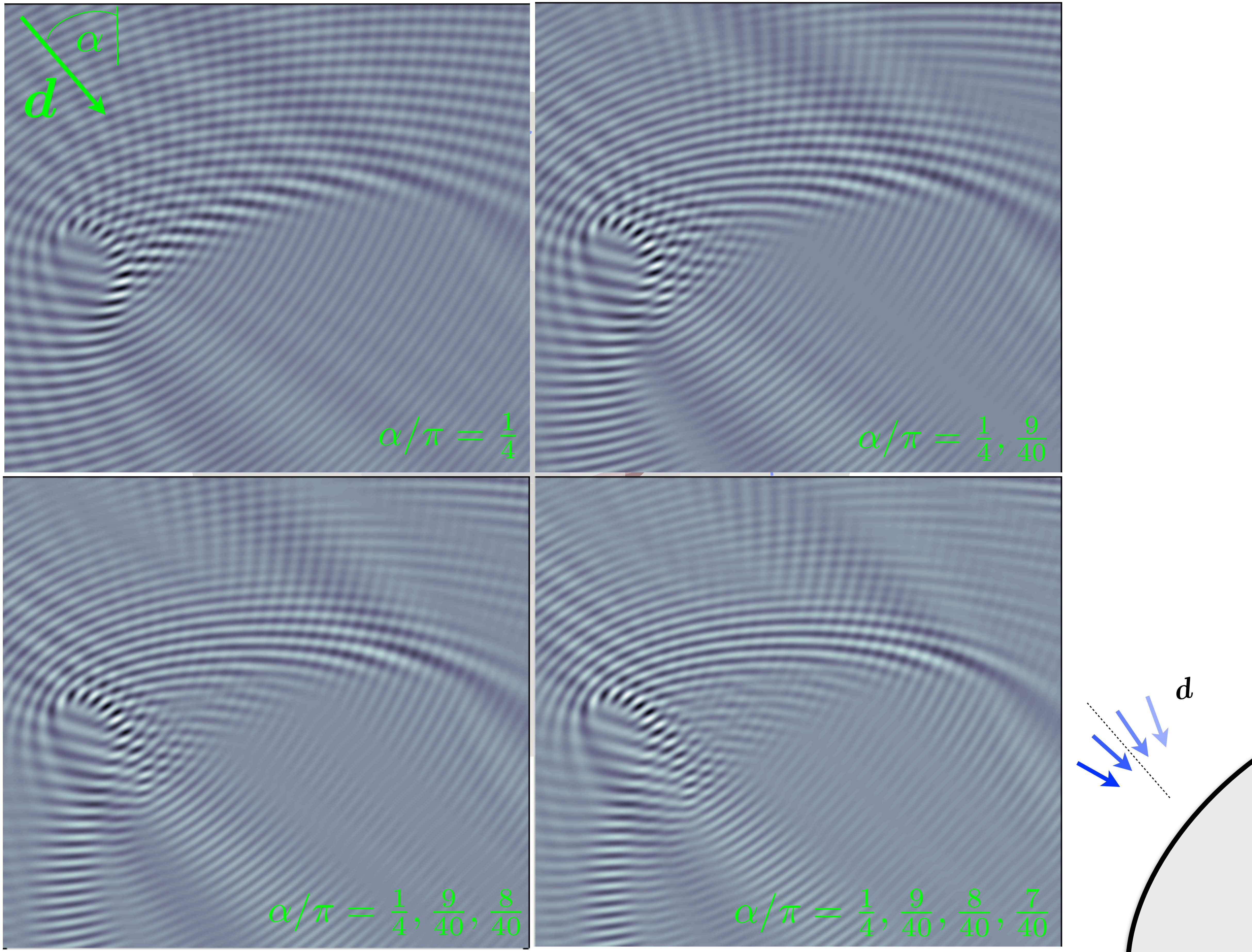


Constructive Interference

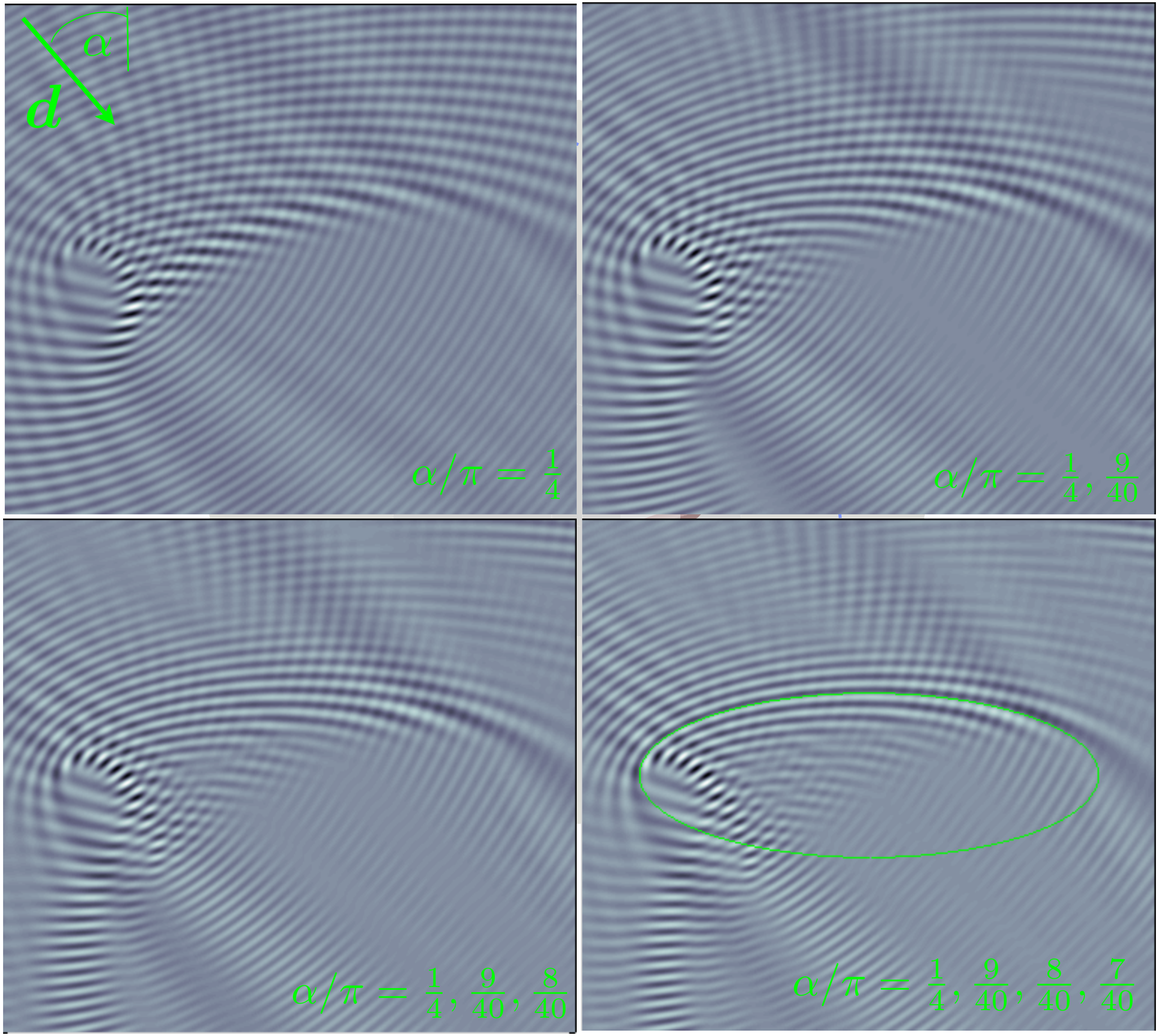


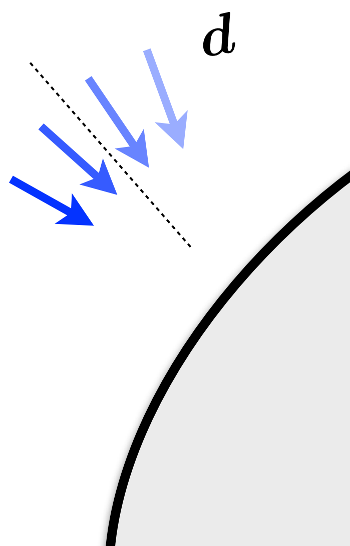


Constructive Interference



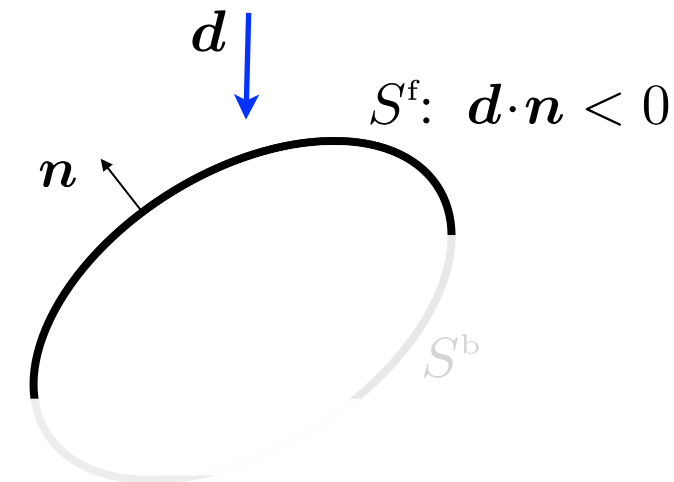
Constructive Interference





Neumann obstacle, large k

Incident plane wave $\quad u^i = e^{-\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{d}}$



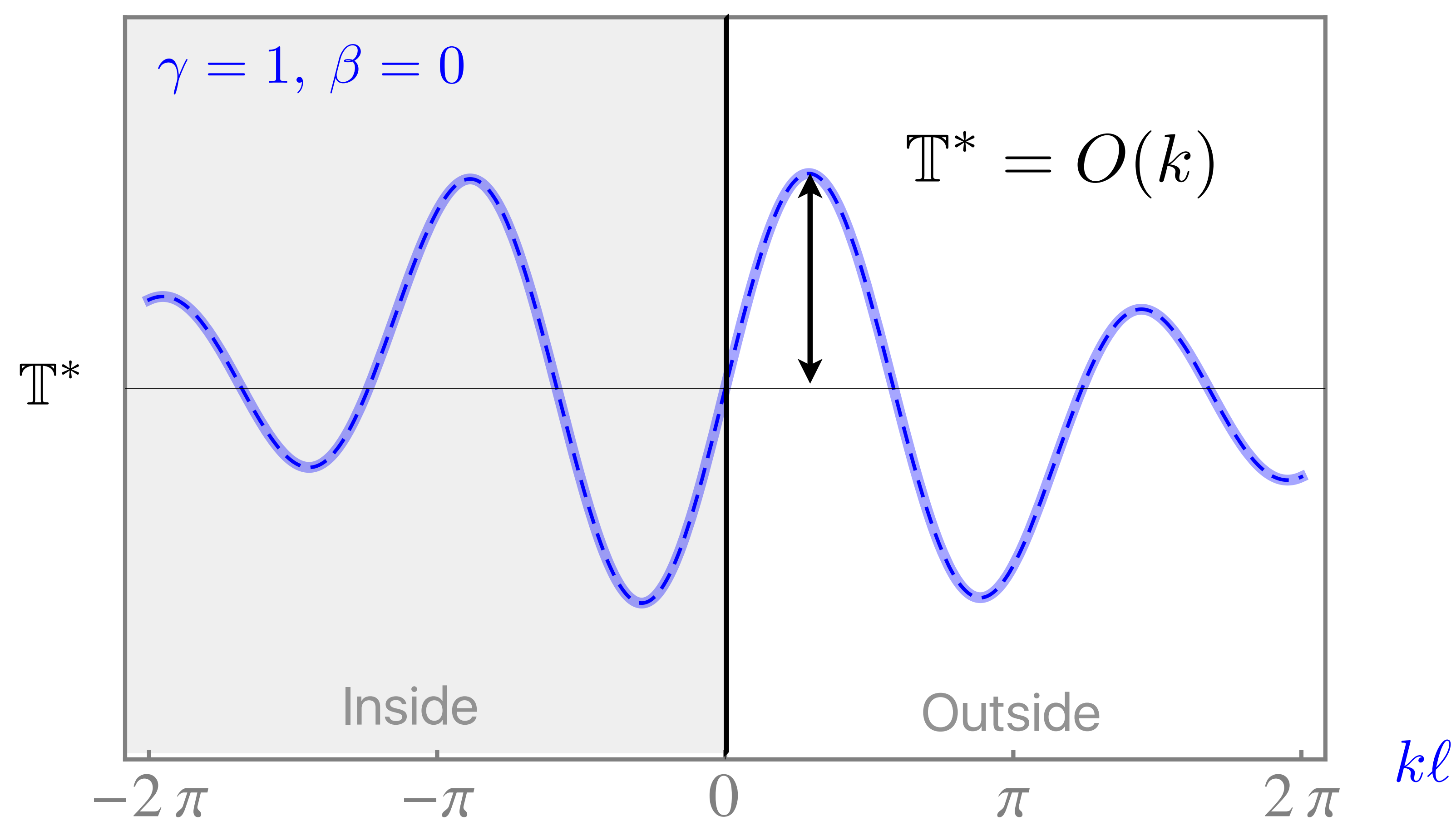
Kirchhoff approximation $\quad kL \gg 1$

$$u = \begin{cases} 2\,u^i & \text{on} \quad S^{\mathrm{f}} \\ 0 & \text{on} \quad S^{\mathrm{b}} \end{cases}, \qquad u_{,n} = 0 \quad \text{on} \quad D,$$

Full aperture

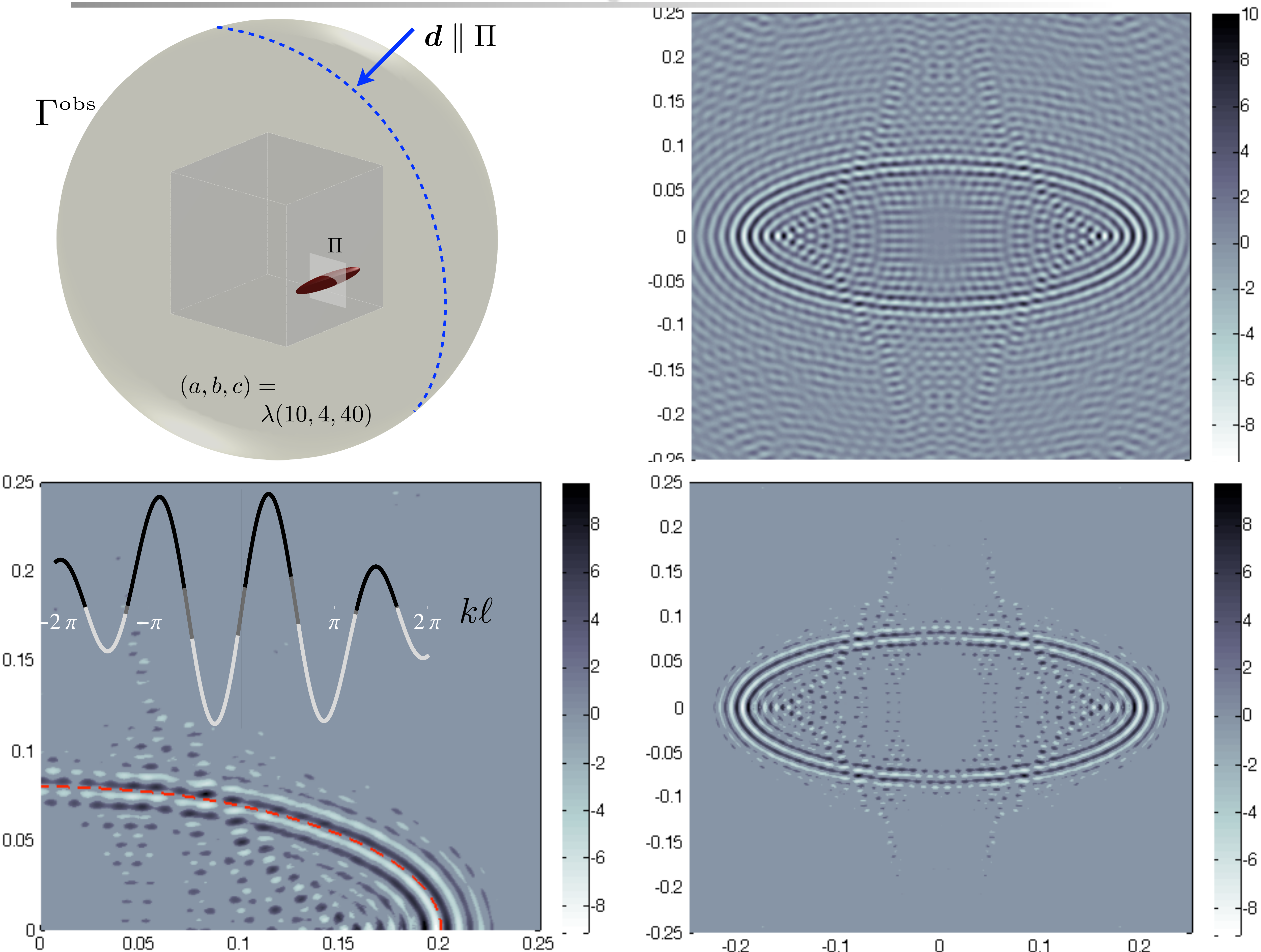
$$\mathbb{T} = \mathbb{T}^* + \mathbb{T}^s \quad (\mathbb{T}^s \to O(1))$$

$$\mathbb{T}^* = \pm \frac{2\pi k}{(k\ell)^3} \left\{ \frac{3(1-\beta)}{2+\beta} (k\ell\cos(k\ell) - \sin(k\ell))^2 + (1-\beta\gamma^2)\,(k\ell)^2\, \sin(k\ell)^2 \right\}$$



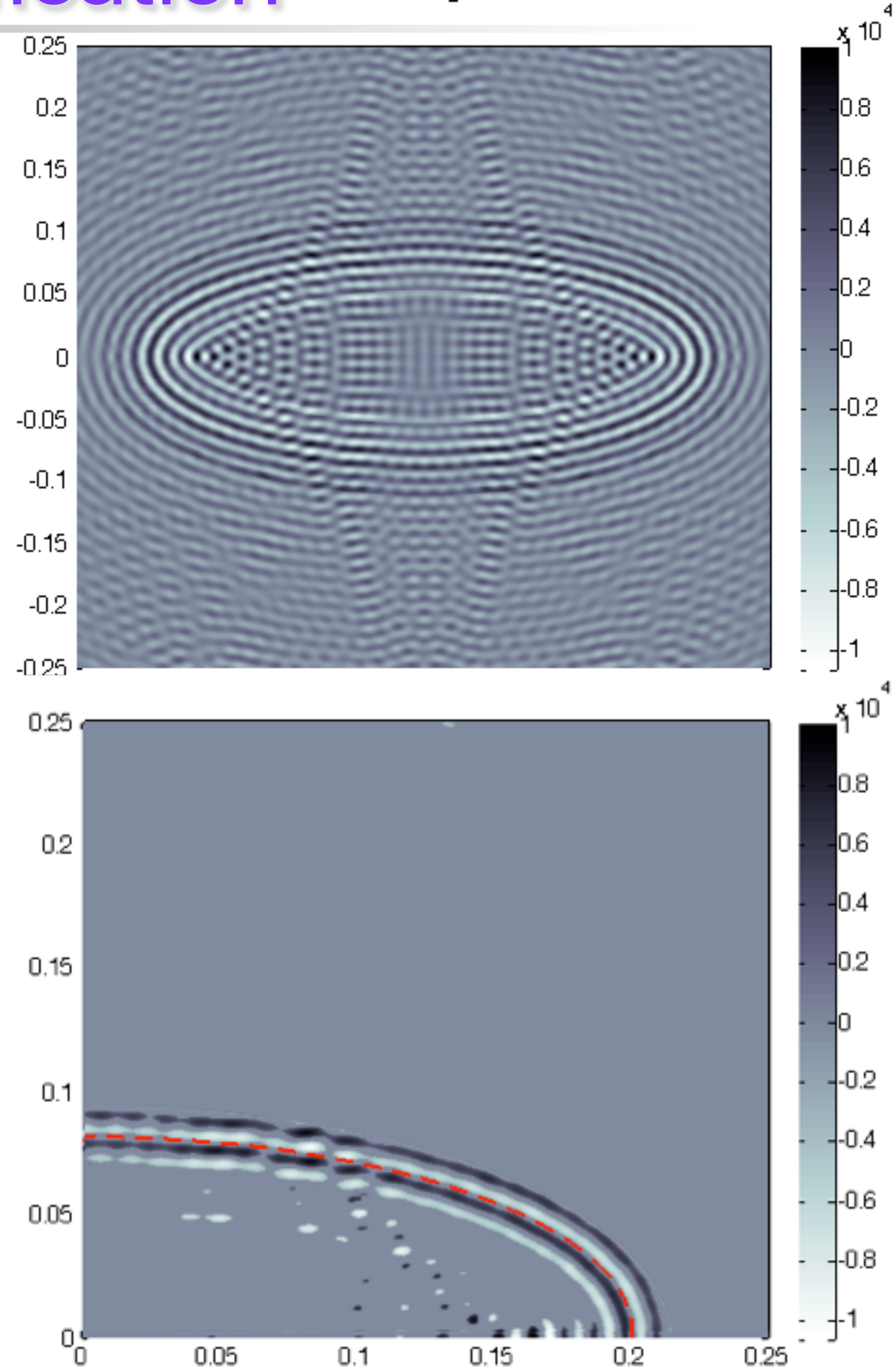
Full aperture

Neumann obstacle



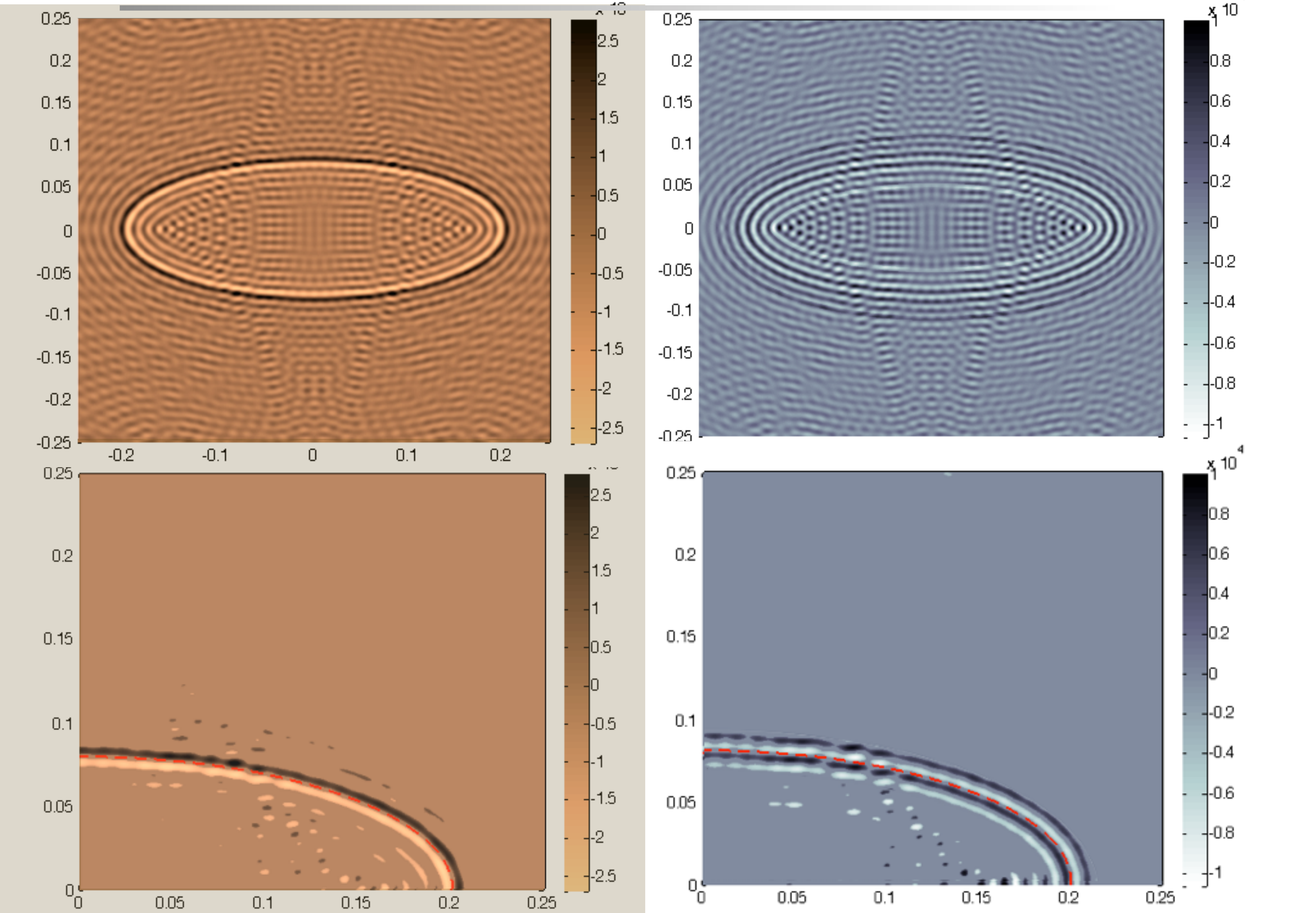
Identification

Dirichlet obstacle



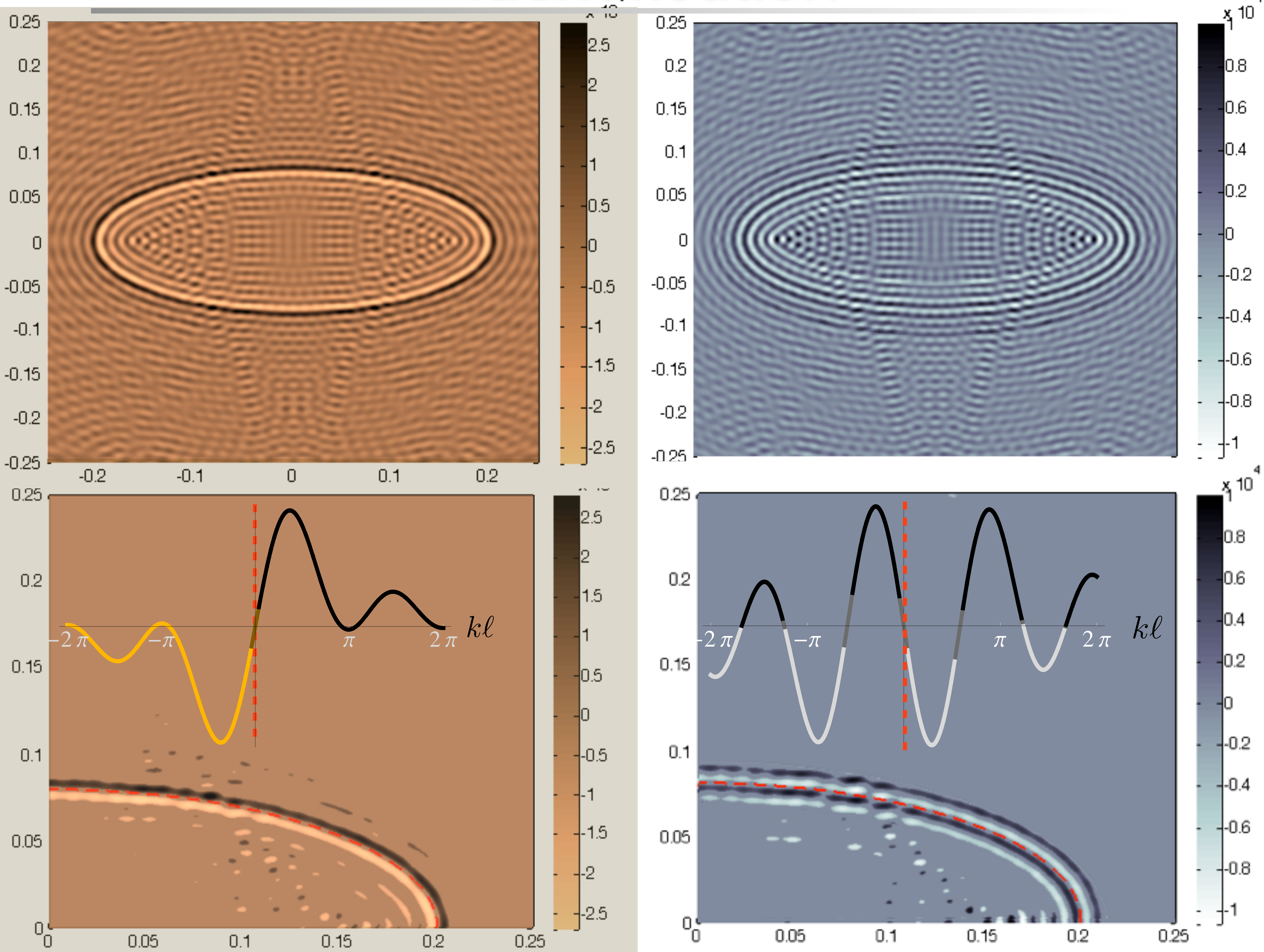
Identification

Dirichlet obstacle



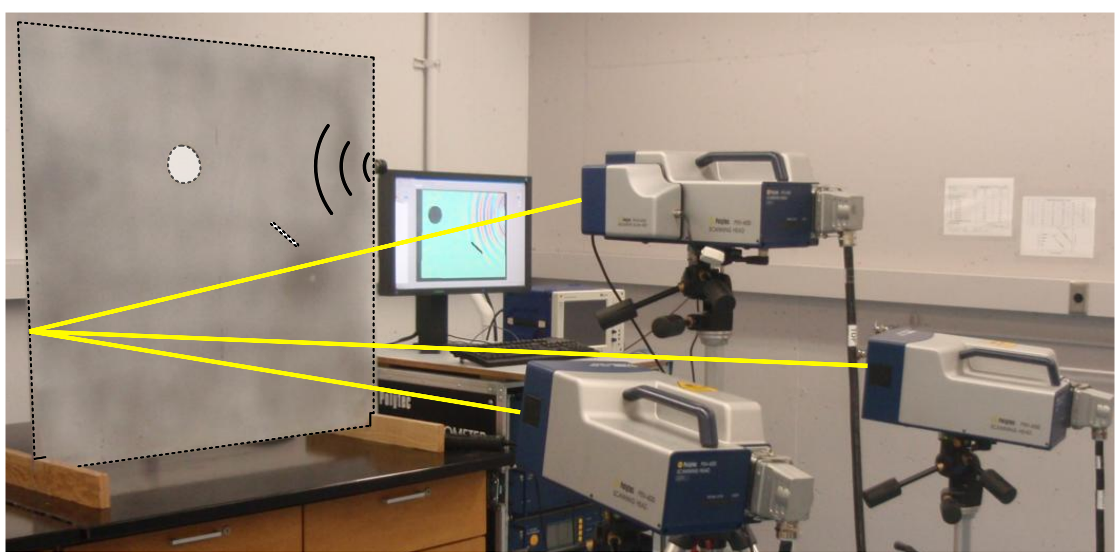
Identification

Dirichlet obstacle

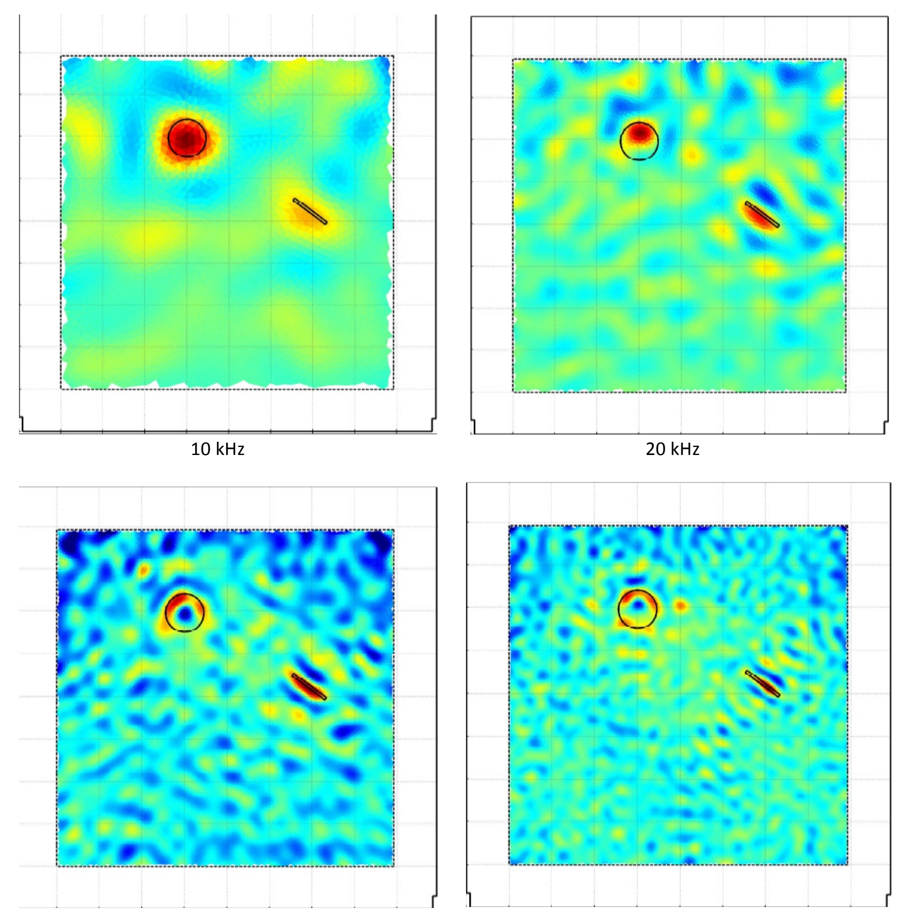


New heuristic

Theory	Application
Helmholtz, $\mathbb{R}^3$	Navier, $\mathbb{R}^2$
Ref. domain: unbounded	Ref. domain: bounded
High frequency: $\lambda \ll L$	Intermediate frequency: $\lambda \simeq L$
Full aperture	Partial aperture



Tokmashev, Tixier & Guzina (2013), *Inverse Problems*



New heuristic

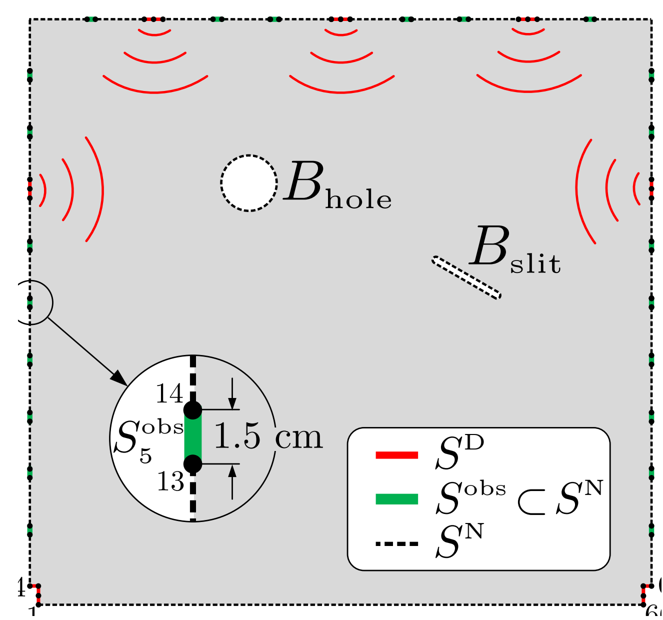
Theory

Helmholtz, $\mathbb{R}^3$

Ref. domain: unbounded

High frequency: $\lambda \ll L$

Full aperture



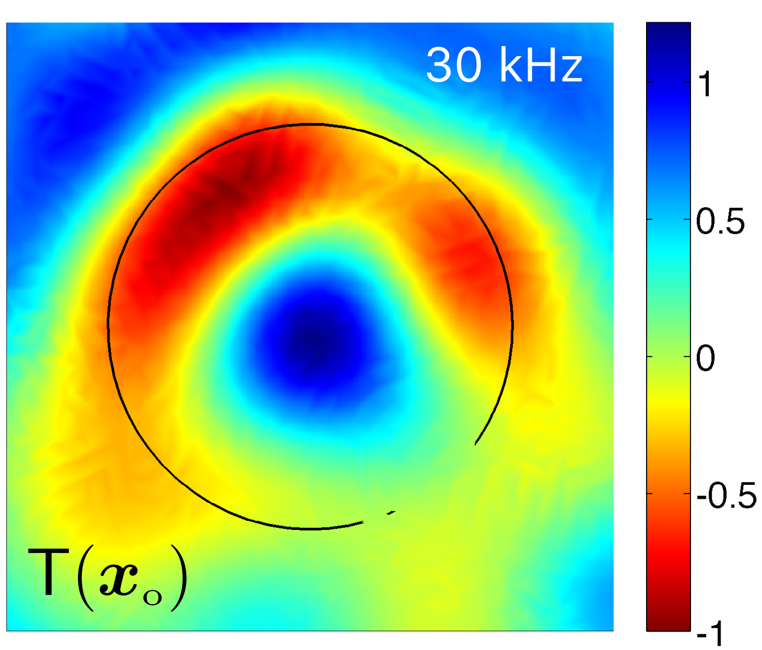
Application

Navier, $\mathbb{R}^2$

Ref. domain: bounded

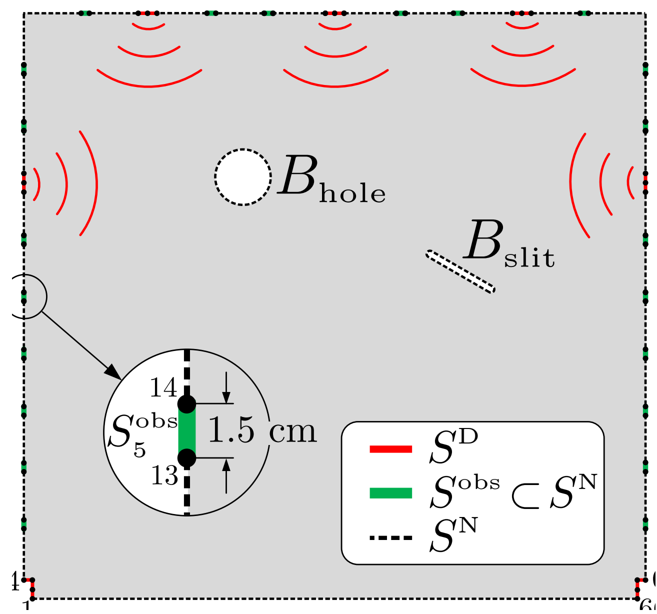
Interm. frequency: $\lambda \simeq L$

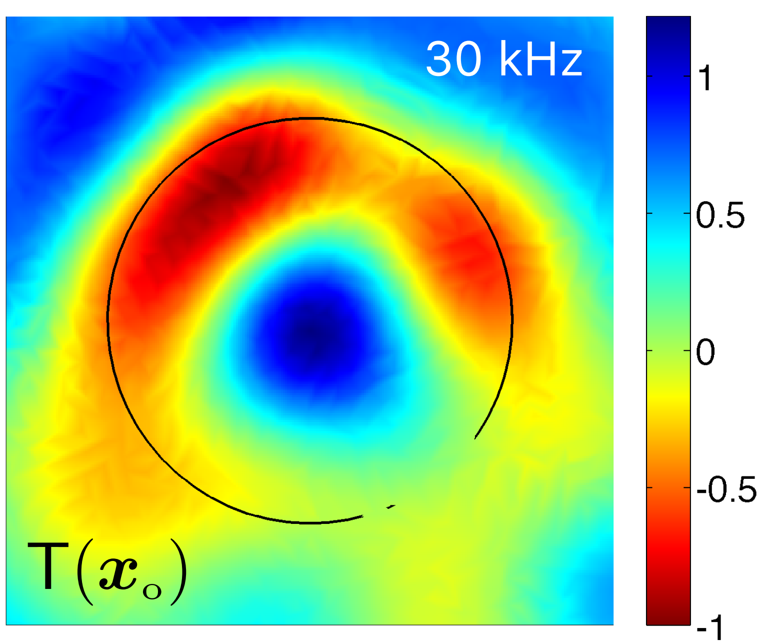
Partial aperture

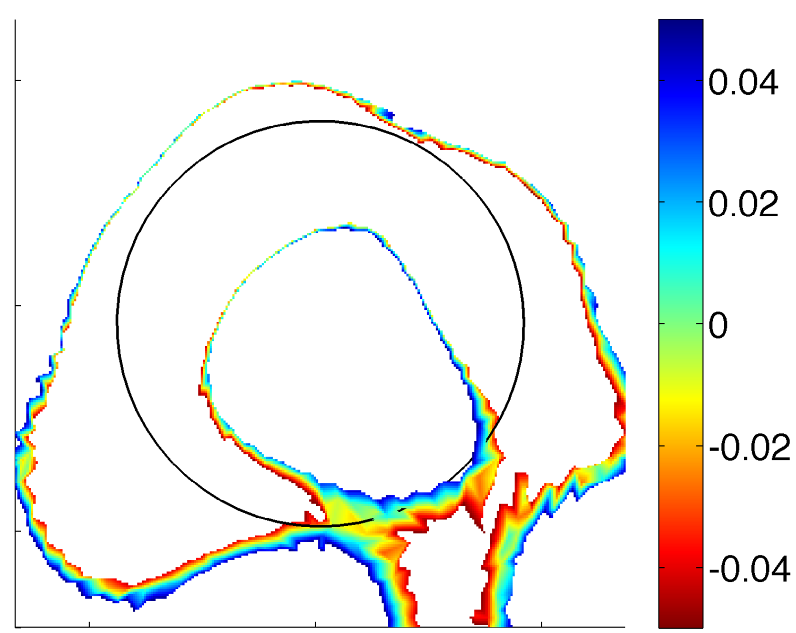


New heuristic

Theory	Application
Helmholtz, $\mathbb{R}^3$	Navier, $\mathbb{R}^2$
Ref. domain: unbounded	Ref. domain: bounded
High frequency: $\lambda \ll L$	Interm. frequency: $\lambda \simeq L$
Full aperture	Partial aperture

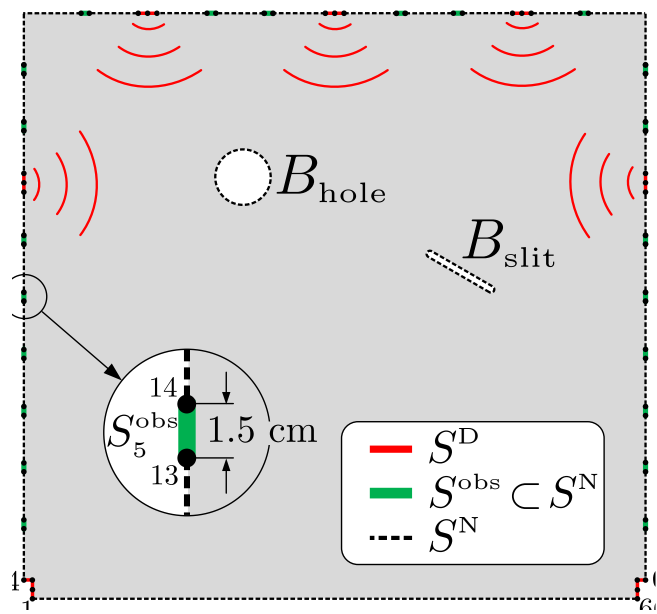


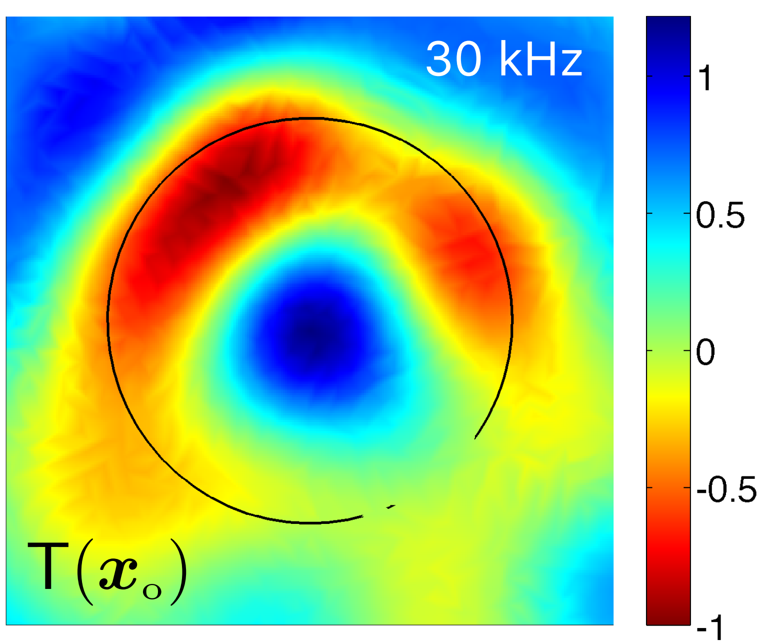


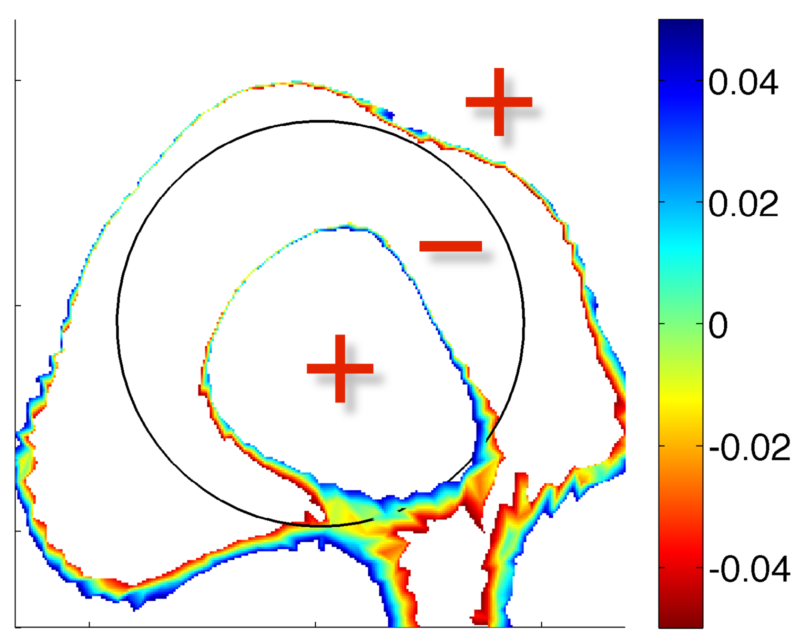


New heuristic

Theory	Application
Helmholtz, $\mathbb{R}^3$	Navier, $\mathbb{R}^2$
Ref. domain: <u>unbounded</u>	Ref. domain: <u>bounded</u>
High frequency: $\lambda \ll L$	Interm. frequency: $\lambda \simeq L$
Full aperture	Partial aperture







New heuristic

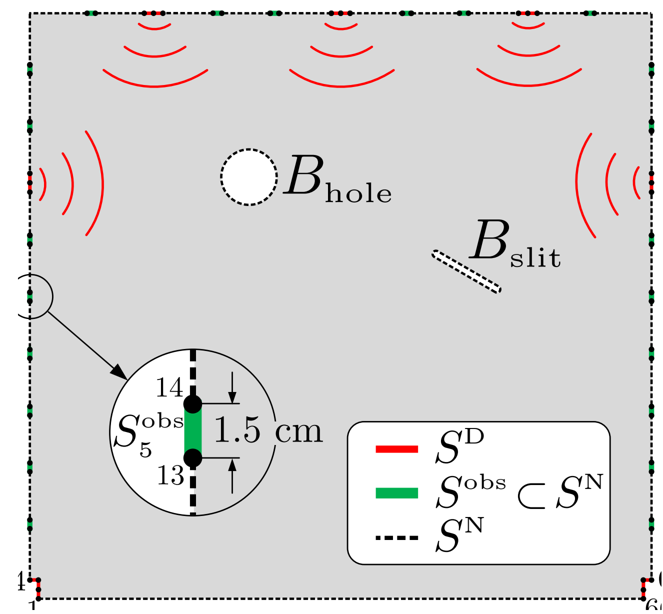
Theory

Helmholtz, $\mathbb{R}^3$

Ref. domain: unbounded

High frequency: $\lambda \ll L$

Full aperture



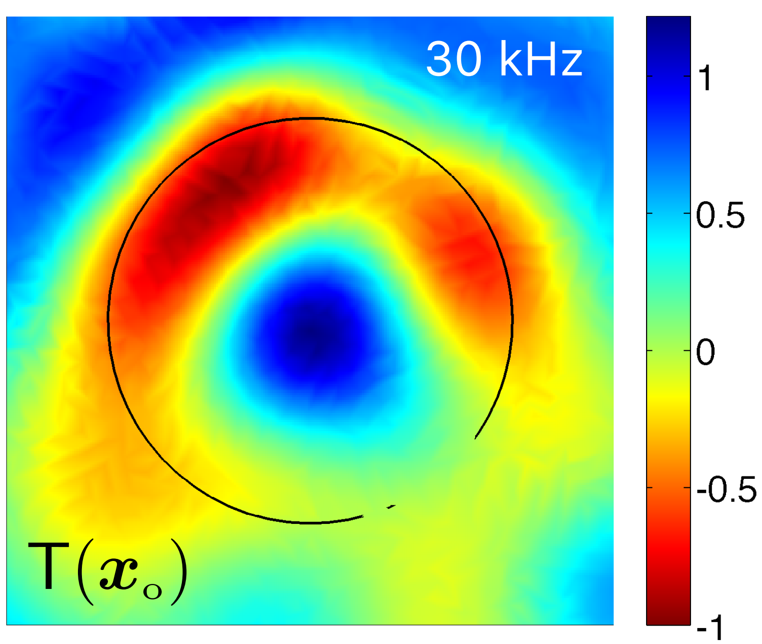
Application

Navier, $\mathbb{R}^2$

Ref. domain: bounded

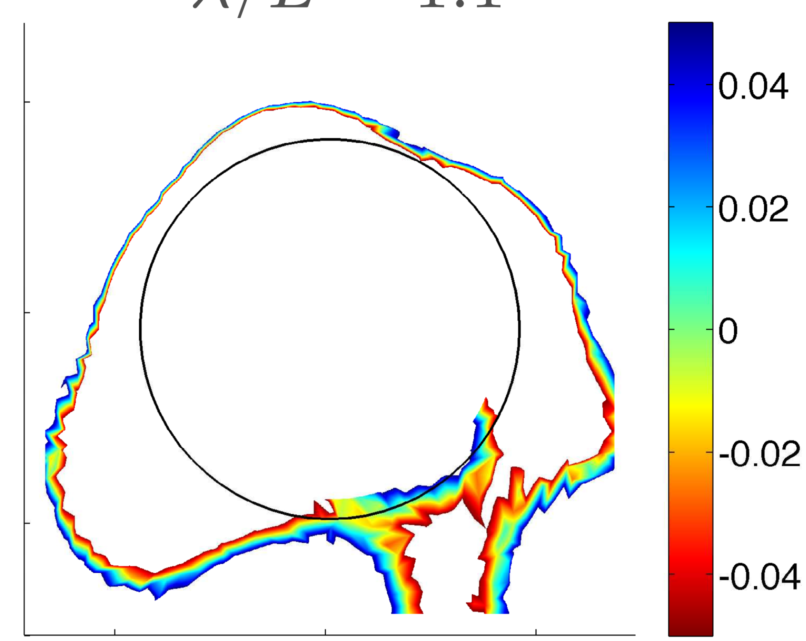
Interm. frequency: $\lambda \simeq L$

Partial aperture



Reconstruction

$\lambda / L = 1.1$



New heuristic

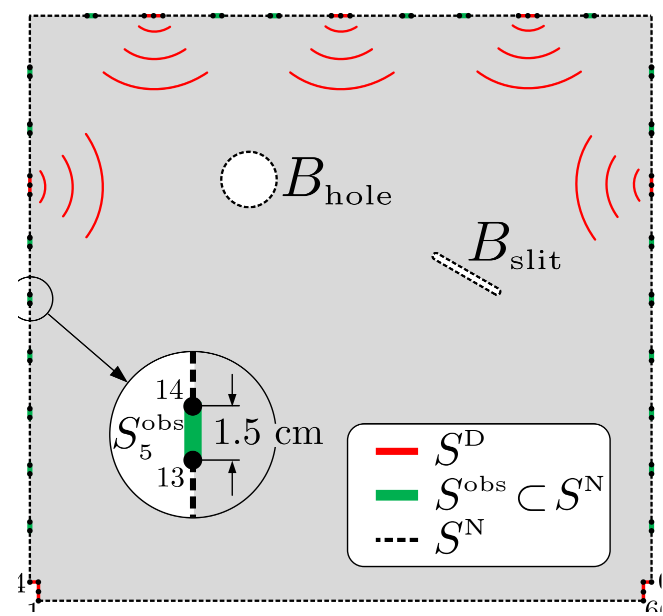
Theory

Helmholtz, $\mathbb{R}^3$

Ref. domain: unbounded

High frequency: $\lambda \ll L$

Full aperture



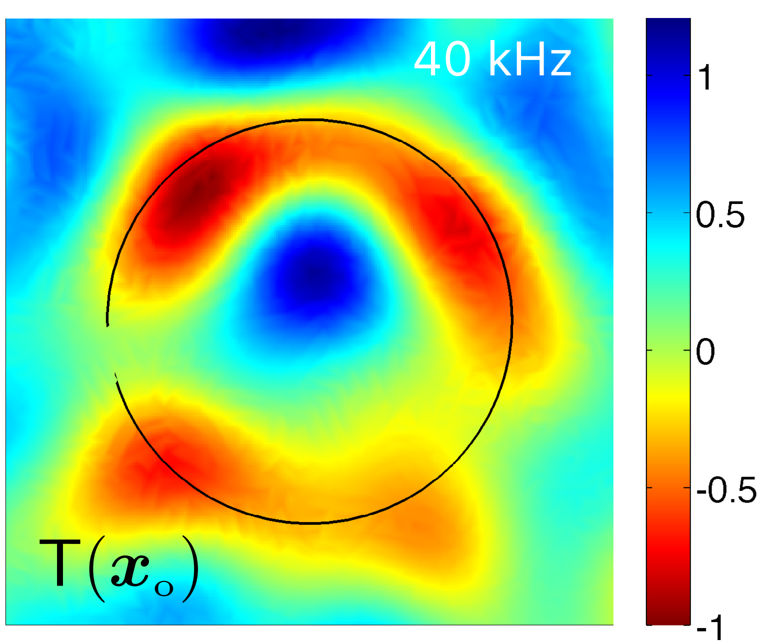
Application

Navier, $\mathbb{R}^2$

Ref. domain: bounded

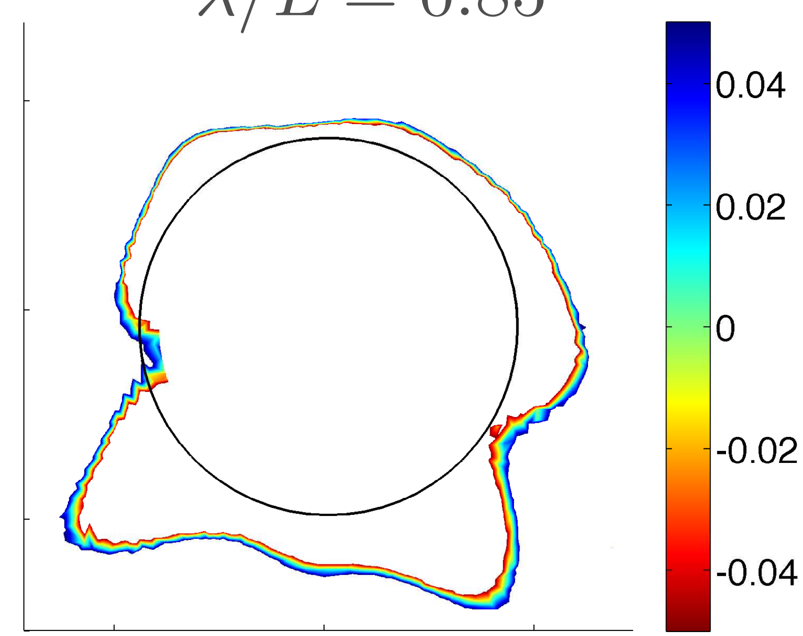
Interm. frequency: $\lambda \simeq L$

Partial aperture



Reconstruction

$\lambda/L = 0.85$



Summary

High-frequency approximation

Convex impenetrable obstacles

Meso/far- field measurements

Non-degenerate	$\Longrightarrow$	$O(k/r)$
Near boundary	$\Longrightarrow$	$O(k)$
Full aperture	$\Longrightarrow$	$O(k/r)$