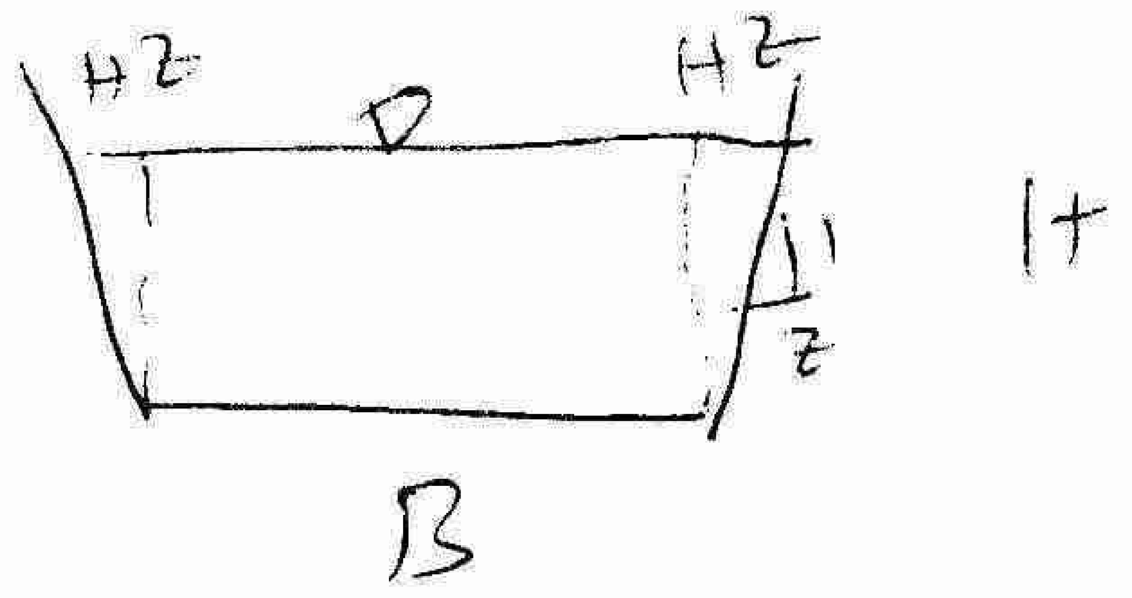
3) (a)



Cross section area $A = BH + 2 \times \frac{1}{2} \times HZ \cdot H$

$= BH + ZH^2$ — (1)

Wetted perimeter $P = B + 2\left(\sqrt{H^2 + H^2 Z^2}\right)$

$P = B + 2H\sqrt{1+Z^2})$ — (2)

from (1), $\frac{A - ZH^2}{H} = B$

Substitute this in (2)

$$P = \frac{A}{\cancel{H}} - ZH + 2H\sqrt{1+Z^2}$$

$$\frac{\partial P}{\partial H} = -\frac{A}{H^2} - Z + 2\sqrt{1+Z^2} = 0$$

$$-\frac{A}{H^2} = Z - 2\sqrt{1+Z^2}$$

Substitute for A from (1)

$$-(BH + ZH^2) = H^2\left(Z - 2\sqrt{1+Z^2}\right)$$

$$B = -HZ + 2H\sqrt{1+Z^2} - ZH$$

$$\boxed{B = 2H\left(\sqrt{1+Z^2} - Z\right)} \quad \text{— (3)}$$

Plug (3) in (2)

$$P = 2H\sqrt{1+Z^2} - 2HZ + 2H\sqrt{1+Z^2}$$

$$P = 4H\cancel{z}\sqrt{1+z^2} - 2Hz \qquad 1+z^2 = x$$

$$P = 2H\cancel{z}\left(2\sqrt{1+z^2} - z\right) \qquad dx = 2z\,dz$$

$$\frac{\partial P}{\partial z} = 0 \Rightarrow \overset{2}{\cancel{4}}H\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{1+z^2}}\right)\cdot 2z - 2H = 0$$

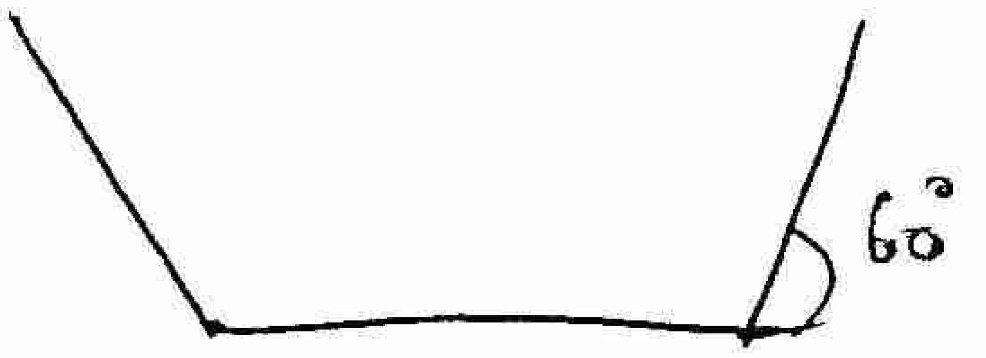
$$\Rightarrow \frac{2z}{\sqrt{1+z^2}} = 1$$

$$4z^2 = 1+z^2$$

$$3z^2 = 1 \Rightarrow z^2 = 1/3$$

$$\boxed{z = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \cot(60)}$$

$\Rightarrow$ the best Trapezoidal Section is a Semi-Hexagon



from ③

$$B = 2H\left(\sqrt{1+\left(\frac{1}{\sqrt{3}}\right)^2} - \frac{1}{\sqrt{3}}\right) = 2H\left(\sqrt{1+\frac{1}{3}} - \frac{1}{\sqrt{3}}\right)$$

$$= 2H\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}H$$

$$\boxed{B = \frac{2}{\sqrt{3}}H}$$

3) (b)

The objective fn to minimize Total Cost is

$$\boxed{C = C_1(A) + C_2(P)}$$

C_1 — Excavation Cost (depends on X-section area)
C_2 — lining cost (depends on wetted perimeter)

$$A = BH + ZH^2 = \left(\frac{2}{\sqrt{3}}H\right)H + \frac{1}{\sqrt{3}} \cdot H^2$$

$$\boxed{A = \sqrt{3}H^2}$$

$$P = B + 2H\sqrt{1+Z^2} = \frac{2}{\sqrt{3}}H + 2H\sqrt{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{2}{\sqrt{3}}H + 2H \cdot \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}(1+2)H = 2H\sqrt{3}$$

$$\boxed{C = C_1\sqrt{3}H^2 + C_2\sqrt{3}\,2H}$$ minimize lining cost

4) From the Graphical approach the approx solution is $(1, 6.5)$

$$C(x, y) = 7.9 + 0.13x + 0.21y - 0.05x^2 - 0.016y^2 - 0.007xy$$

$$\frac{\partial C}{\partial x} = 0.13 - 0.1x - 0.007y = 0 \quad \text{—} ①$$

$$\frac{\partial C}{\partial y} = 0.21 - 0.032y - 0.007x = 0 \quad \text{—} ②$$

Solving ① & ② yields: $(0.854, 6.376)$
very close to the Graphical solution.

$$\frac{\partial^2 C}{\partial x^2} = -0.1, \quad \frac{\partial^2 C}{\partial y^2} = -0.032$$

$$\frac{\partial^2 C}{\partial x \partial y} = -0.007, \quad \frac{\partial^2 C}{\partial y \partial x} = -0.007$$

$|H|$ = determinant of the Hessian matrix

$$= \frac{\partial^2 f}{\partial x^2 \partial y^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= (-0.1)(-0.032) - (-0.007)^2 = 0.003151$$

$|H| > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ $\Rightarrow$ the Solution we obtained is the local maximum