Lecture Notes in:

Mechanics and Design of

REINFORCED CONCRETE

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Chapter 1

INTRODUCTION

1.1 Material

1.1.1 Concrete

This section is adapted from Concrete by Mindess and Young, Prentice Hall, 1981

1.1.1.1 Mix Design

1.1.1.1.1 Constituents

Concrete is a mixture of Portland cement, water, and aggregates (usually sand and crushed stone).

Portland cement is a mixture of calcareous and argillaceous materials which are calcined in a kiln and then pulverized. When mixed with water, cement hardens through a process called hydration.

Ideal mixture is one in which:

1. A minimum amount of cement-water paste is used to fill the interstices between the particles of aggregates.

2. A minimum amount of water is provided to complete the chemical reaction with cement. Strictly speaking, a water/cement ratio of about 0.25 is needed to complete this reaction, but then the concrete will have a very low “workability”.

In such a mixture, about 3/4 of the volume is constituted by the aggregates, and the remaining 1/4 being the cement paste.

Smaller particles up to 1/4 in. in size are called fine aggregates, and the larger ones being coarse aggregates.

Portland Cement has the following ASTM designation

I Normal

II Moderate sulfate resistant, moderate heat of hydration

III High early strength (but releases too much heat)
### Table 1.1: ASTM Sieve Designation’s Nominal Sizes Used for Concrete Aggregates

<table>
<thead>
<tr>
<th>Sieve Size</th>
<th>% Passing Each Sieve (Nominal Maximum Size)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1/2 in.</td>
<td>95-100 100 - -</td>
</tr>
<tr>
<td>1 in.</td>
<td>- 95-100 100 -</td>
</tr>
<tr>
<td>3/4 in.</td>
<td>35-70 - 90-100 100</td>
</tr>
<tr>
<td>1/2 in.</td>
<td>- 25-60 - 90-100</td>
</tr>
<tr>
<td>3/8 in.</td>
<td>10-30 - 20-55 40-70</td>
</tr>
<tr>
<td>No. 4</td>
<td>0-5 0-10 0-10 0-15</td>
</tr>
<tr>
<td>No. 8</td>
<td>- 0-5 0-5 0-5</td>
</tr>
</tbody>
</table>

### Table 1.2: ASTM C33 Grading Limits for Coarse Concrete Aggregates

<table>
<thead>
<tr>
<th>Sieve Size</th>
<th>% Passing</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/4 in.</td>
<td>100</td>
</tr>
<tr>
<td>No. 4</td>
<td>95-100</td>
</tr>
<tr>
<td>No. 8</td>
<td>80-100</td>
</tr>
<tr>
<td>No. 16</td>
<td>50-85</td>
</tr>
<tr>
<td>No. 30</td>
<td>25-60</td>
</tr>
<tr>
<td>No. 50</td>
<td>10-30</td>
</tr>
<tr>
<td>No. 100</td>
<td>2-10</td>
</tr>
</tbody>
</table>

### Table 1.3: ASTM C33 Grading Limits for Fine Concrete Aggregates
### Table 1.4: Example of Fineness Modulus Determination for Fine Aggregate

<table>
<thead>
<tr>
<th>Sieve Size</th>
<th>Weight Retained (g)</th>
<th>Amount Retained (wt. %)</th>
<th>Cumulative Amount Retained (%)</th>
<th>Cumulative Amount Passing (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 4</td>
<td>9</td>
<td>2</td>
<td>2</td>
<td>98</td>
</tr>
<tr>
<td>No. 8</td>
<td>46</td>
<td>9</td>
<td>11</td>
<td>89</td>
</tr>
<tr>
<td>No. 16</td>
<td>97</td>
<td>19</td>
<td>30</td>
<td>70</td>
</tr>
<tr>
<td>No. 30</td>
<td>99</td>
<td>20</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>No. 50</td>
<td>120</td>
<td>24</td>
<td>74</td>
<td>26</td>
</tr>
<tr>
<td>No. 100</td>
<td>91</td>
<td>18</td>
<td>92</td>
<td>8</td>
</tr>
<tr>
<td>Sample Weight 500 g.</td>
<td></td>
<td></td>
<td>$\sum = 259$</td>
<td></td>
</tr>
<tr>
<td>Fineness modulus $= \frac{259}{100} = 2.59$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1.1.1.1.3 Mix procedure

Before starting the mix design process, the following **material properties** should be determined:

1. Sieve analysis of both fine and coarse aggregates
2. Unit weight of the coarse aggregate
3. Bulk specific gravities
4. absorption capacities of the aggregates
### Table 1.7: Approximate Mixing Water Requirements, lb/yd$^3$ of Concrete For Different Slumps and Nominal Maximum Sizes of Aggregates

<table>
<thead>
<tr>
<th>Slump in.</th>
<th>Sizes of Aggregates</th>
<th>Non-Air-Entrained Concrete</th>
<th>Air-Entrained Concrete</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3/8 in.</td>
<td>1/2 in.</td>
<td>3/4 in.</td>
</tr>
<tr>
<td>1-2</td>
<td>350</td>
<td>335</td>
<td>315</td>
</tr>
<tr>
<td>3-4</td>
<td>385</td>
<td>365</td>
<td>340</td>
</tr>
<tr>
<td>6-7</td>
<td>410</td>
<td>385</td>
<td>360</td>
</tr>
</tbody>
</table>

### Table 1.8: Relationship Between Water/Cement Ratio and Compressive Strength

<table>
<thead>
<tr>
<th>28 days $f'_c$</th>
<th>$w/c$ Ratio by Weight</th>
<th>Non-air-entrained</th>
<th>Air-entrained</th>
</tr>
</thead>
<tbody>
<tr>
<td>6,000</td>
<td>0.41</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5,000</td>
<td>0.48</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>4,000</td>
<td>0.57</td>
<td>0.48</td>
<td></td>
</tr>
<tr>
<td>3,000</td>
<td>0.68</td>
<td>0.59</td>
<td></td>
</tr>
<tr>
<td>2,000</td>
<td>0.82</td>
<td>0.74</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.8: Relationship Between Water/Cement Ratio and Compressive Strength
**Fine Aggregates:** Bulk specific gravity (SSD) = 2.65; absorption capacity = 1.3 %; Total moisture content=5.5%; fineness modulus = 2.70

The sieve analyses of both the coarse and fine aggregates fall within the specified limits. With this information, the mix design can proceed:

1. **Choice of slump** is consistent with Table 1.5.

2. **Maximum aggregate size** (3/4 in) is governed by reinforcing details.

3. Estimation of mixing **water:** Because water will be exposed to freeze and thaw, it must be air-entrained. From Table 1.6 the air content recommended for extreme exposure is 6.0%, and from Table 1.7 the water requirement is 280 lb/yd$^3$.

4. From Table 1.8, the **water to cement ratio** estimate is 0.4.

5. **Cement content,** based on steps 4 and 5 is $280/0.4=700$ lb/yd$^3$.

6. **Coarse aggregate content,** interpolating from Table 1.9 for the fineness modulus of the fine aggregate of 2.70, the volume of dry-rodded coarse aggregate per unit volume of concrete is 0.63. Therefore, the coarse aggregate will occupy $0.63 \times 27 = 17.01$ ft$^3$/yd$^3$. The OD weight of the coarse aggregate is $17.01$ ft$^3$/yd$^3 \times 100$ lbs/ft$^3=1,701$ lb. The SSD weight is $1,701 \times 1.01=1,718$ lb.

7. **Fine aggregate content** Knowing the weights and specific gravities of the water, cement, and coarse aggregate, and knowing the air volume, we can calculate the volume per yd$^3$ occupied by the different ingredients.

   - Water: $280/62.4 = 4.49$ ft$^3$
   - Cement: $700/(3.15)(62.4) = 3.56$ ft$^3$
   - Coarse Aggregate (SSD): $1,718/(2.70)(62.4) = 1.62$ ft$^3$
   - Air: $0.06(27) = 1.62$ ft$^3$

   Hence, the fine aggregate must occupy a volume of $27.0 - 19.87 = 7.13$ ft$^3$. The required SSD weight of the fine aggregate is $7.13$ ft$^3 \times (2.65)/(62.4)$lb/ft$^3 = 1,179$ lbs lb.

8. **Adjustment for moisture** in the aggregate. Since the aggregate will be neither SSD or OD in the field, it is necessary to adjust the aggregate weights for the amount of water contained in the aggregate. Only surface water need be considered; absorbed water does not become part of the mix water. For the given moisture contents, the adjusted aggregate weights become:

   - Coarse aggregate (wet): $1,718(1.025-0.01) = 1,744$ lb/yd$^3$ of dry coarse
   - Fine aggregate (wet): $1,179(1.055-0.013) = 1,229$ lb/yd$^3$ of dry fine

   Surface moisture contributed by the coarse aggregate is $2.5-1.0 = 1.5$%; by the fine aggregate: $5.5-1.3 = 4.2$%; Hence we need to decrease water to $280-1,718(0.015)-1,179(0.042) = 205$ lb/yd$^3$.

   Thus, the estimated batch weight per yd$^3$ are
Chapter 2

FLEXURE

1. This is probably the longest chapter in the notes, we shall cover in great details flexural design/analysis of R/C beams starting with uncracked section to failure conditions.

1. Uncracked elastic (uneconomical)
2. Cracked elastic (service stage)
3. Ultimate (failure)

2.1 Uncracked Section

Assuming perfect bond between steel and concrete, we have $\varepsilon_s = \varepsilon_c$, Fig. 2.1

$$\varepsilon_s = \varepsilon_c \Rightarrow \frac{f_s}{E_s} = \frac{f_c}{E_c} \Rightarrow f_s = \frac{E_s}{E_c} f_c \Rightarrow f_s = n f_c$$  \hspace{1cm} (2.1)

where $n$ is the modular ratio $n = \frac{E_s}{E_c}$

3. Tensile force in steel $T_s = A_s f_s = A_s n f_c$

4. Replace steel by an equivalent area of concrete, Fig. 2.2.

Figure 2.1: Strain Diagram Uncracked Section
2.2 Section Cracked, Stresses Elastic

This is important not only as an acceptable alternative ACI design method, but also for the later evaluation of crack width under service loads.

2.2.1 Basic Relations

If \( f_{ct} > f_r \), \( f_{cc} \approx 0.5f_c \) and \( f_s < f_y \) we will assume that the crack goes all the way to the N.A and we will use the transformed section, Fig. 2.3

\[
f_{ct} = \frac{Mc}{I} = \frac{(540,000) \text{ lb.in}(25 - 13.2) \text{ in}^4}{(14,722) \text{ in}^4} = 433 \text{ psi} < 475 \text{ psi} \quad (2.3-h)
\]

\[
f_s = n \frac{Mc}{I} = (8)\frac{(540,000)(23 - 13.2) \text{ in}^4}{(14,722)} = 2,876 \text{ psi} \quad (2.3-i)
\]

Figure 2.3: Stress Diagram Cracked Elastic Section

To locate N.A, tension force = compressive force (by def. NA) (Note, for linear stress distribution and with \( \Sigma F_x = 0 \); \( \sigma = by \) \( \Rightarrow \int bydA = 0 \), thus \( b \int ydA = 0 \) and \( \int ydA = \frac{\pi}{2}A = 0 \), by definition, gives the location of the neutral axis)

Note, N.A. location depends only on geometry & \( n \left( \frac{E_s}{E_c} \right) \)

Tensile and compressive forces are equal to \( C = \frac{bd}{2}f_c \) & \( T = A_s f_s \) and neutral axis is determined by equating the moment of the tension area to the moment of the compression area

\[
b(kd)\left(\frac{kd}{2}\right) = nA_s(d - kd) \quad 2^{nd} \text{ degree equation} \quad (2.4-a)
\]

\[
M = Tjd = A_s f_s jd \Rightarrow f_s = \frac{M}{A_s jd} \quad (2.4-b)
\]

\[
M = Cjd = \frac{bd}{2}f_c jd = \frac{bd^2}{2}k_j f_c \Rightarrow f_c = \frac{M}{\frac{bd^2}{2}k_j} \quad (2.4-c)
\]

where \( j = (1 - k/3) \).
2.2 Section Cracked, Stresses Elastic

**Review** Start by determining $\rho$,

- If $\rho < \rho_b$ steel reaches max. allowable value before concrete, and
  \[
  M = A_s f_s j d = A_s f_s j d \\
  \text{(2.9)}
  \]

- If $\rho > \rho_b$ concrete reaches max. allowable value before steel and
  \[
  M = f_c b k d = \frac{1}{2} f_c b k d \\
  \text{or}
  M = \frac{1}{2} f_c j k b d = R b d \\
  \text{(2.11)}
  \]

where

\[
\begin{align*}
  k &= \sqrt{2 \rho n + (\rho n)^2} - \rho n \\
  \end{align*}
  \]

**Design** We define

\[
R \overset{\text{def}}{=} \frac{1}{2} f_c k j
\]

where $k = \frac{n}{n + r}$, solve for $b d^2$ from

\[
bd^2 = \frac{M}{R} \\
\text{(2.13)}
\]

assume $b$ and solve for $d$. Finally we can determine $A_s$ from

\[
A_s = \rho b d \\
\text{(2.14)}
\]

**Summary**

<table>
<thead>
<tr>
<th>Review</th>
<th>Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b, d, A_s \sqrt{M}$</td>
<td>$M \sqrt{b, d, A_s}$</td>
</tr>
<tr>
<td>$\rho = \frac{A_s}{b d}$</td>
<td>$k = \frac{n}{n + r}$</td>
</tr>
<tr>
<td>$k = \sqrt{2 \rho n + (\rho n)^2} - \rho n$</td>
<td>$j = 1 - \frac{k}{3}$</td>
</tr>
<tr>
<td>$r = \frac{f_c}{f_s}$</td>
<td>$r = \frac{f_c}{f_s}$</td>
</tr>
<tr>
<td>$\rho_b = \frac{n}{2r(n + r)}$</td>
<td>$R = \frac{1}{2} f_c k j$</td>
</tr>
<tr>
<td>$\rho &lt; \rho_b$</td>
<td>$\rho_b = \frac{n}{2r(n + r)}$</td>
</tr>
<tr>
<td>$M = A_s f_s j d$</td>
<td>$bd^2 = \frac{M}{R}$</td>
</tr>
<tr>
<td>$\rho &gt; \rho_b$</td>
<td>$A_s = \rho_b b d$ or $A_s = \frac{M}{f_s j d}$</td>
</tr>
</tbody>
</table>

**Example 2-2: Cracked Elastic Section**
Solution:

\[
\rho = \frac{A_s}{bd} = \frac{2.35}{(10)(23)} = .0102 \tag{2.16-a}
\]

\[f_s = 24 \text{ ksi} \tag{2.16-b}\]

\[f_c = (.45)(4,000) = 1,800 \text{ psi} \tag{2.16-c}\]

\[k = \sqrt{2pm + (pn)^2 - pm} = \sqrt{2(.0102)8 + (.0102)^2 - (8)(.0102)} = .331 \tag{2.16-d}\]

\[j = 1 - \frac{k}{3} = .889 \tag{2.16-e}\]

\[N.A. \at \left(.331\right)(23) = 7.61 \text{ in} \tag{2.16-f}\]

\[\rho_b = \frac{n}{2r(n + r)} = \frac{8}{(2)(13.33)(8 + 13.33)} = .014 > \rho \Rightarrow \text{Steel reaches elastic} \tag{2.16-g}\]

\[M = A_s f_s j d = (2.35)(24)(.889)(23) = \boxed{1,154 \text{ k.in} = 96 \text{ k.ft}} \tag{2.16-h}\]

Note, had we used the alternate equation for moment (wrong) we would have overestimated the design moment:

\[M = \frac{1}{2} f_c b k d^2 j \tag{2.17-a}\]

\[= \frac{1}{2} (1.8)(10)(0.33)(0.89)(23)^2 = 1,397 \text{ k.in} > 1,154 \text{ k.in} \tag{2.17-b}\]

If we define \(\alpha_c = f_c/1,500 \) and \(\alpha_s = f_s/24,000\), then as the load increases both \(\alpha_c\) and \(\alpha_s\) increase, but at different rates, one of them \(\alpha_s\) reaches 1 before the other.

\[\alpha_s \quad \alpha_c\]

\[\text{Load}\]

\[\boxed{\text{Example 2-4: Working Stress Design Method; Design}}\]

Design a beam to carry \(LL = 1.9 \text{ k/ft}, DL = 1.0 \text{ k/ft}\) with \(f'_c = 4,000 \text{ psi} \), \(f_y = 60,000 \text{ psi}\), \(L = 32 \text{ ft}\).
2.3 Cracked Section, Ultimate Strength Design Method

\[ \alpha = \frac{f_{av}}{f'_c} \]  
\[ a = \beta_1 c \]  

Thus

\[ \gamma = \frac{\alpha}{\beta_1} \]  

But the location of the resultant forces must be the same, hence

\[ \beta_1 = 2\beta \]  

From Experiments

<table>
<thead>
<tr>
<th>( f'_c ) (psi)</th>
<th>&lt;4,000</th>
<th>5,000</th>
<th>6,000</th>
<th>7,000</th>
<th>8,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>.72</td>
<td>.68</td>
<td>.64</td>
<td>.60</td>
<td>.56</td>
</tr>
<tr>
<td>( \beta )</td>
<td>.425</td>
<td>.400</td>
<td>.375</td>
<td>.350</td>
<td>.325</td>
</tr>
<tr>
<td>( \beta_1 = 2\beta )</td>
<td>.85</td>
<td>.80</td>
<td>.75</td>
<td>.70</td>
<td>.65</td>
</tr>
<tr>
<td>( \gamma = \alpha/\beta_1 )</td>
<td>0.85</td>
<td>0.85</td>
<td>0.85</td>
<td>0.86</td>
<td>0.86</td>
</tr>
</tbody>
</table>

Thus we have, (ACI-318 10.2.7.3):

\[ \beta_1 = .85 \]
\[ = .85 - (.05)(f'_c - 4,000) \left\{ \begin{array}{ll} \frac{1}{1,000} & \text{if } f'_c \leq 4,000 \\ \frac{1}{f'_c - 4,000} & \text{if } 4,000 < f'_c < 8,000 \end{array} \right. \]  

Failure can occur by either

yielding of steel: \( \varepsilon_s = \varepsilon_y \); Progressive

crushing of concrete: \( \varepsilon_c = .003 \); Sudden; (ACI 10.3.2).
2.3 Cracked Section, Ultimate Strength Design Method

Also we need to specify a minimum reinforcement ratio

\[
\rho_{\text{min}} \geq \frac{200}{f_y} \quad (\text{ACI 10.5.1})
\]  

\[ (2.29) \]

to account for temperature & shrinkage

Note, that \( \rho \) need not be as high as 0.75\( \rho_b \). If steel is relatively expensive, or deflection is of concern, can use lower \( \rho \).

As a rule of thumb, if \( \rho < 0.5\rho_b \), there is no need to check for deflection.

2.3.3 Review

Given, \( b, d, A_s, f'_c, f_y \), determine the moment capacity \( M \).

\[
\rho_{\text{act}} = \frac{A_s}{bd} \quad \rho_b = \frac{(0.85)\beta_1 f'_c}{f_y} \quad \frac{85}{87+f_y}
\]

\[ (2.30) \]

- \( \rho_{\text{act}} < \rho_b \): Failure by yielding and

\[
\begin{align*}
\alpha &= \frac{A_s f_y}{0.85 f'_c} \\
M_d &= \phi A_s f_y \left( d - \frac{a}{2} \right) \quad \Sigma M = 0
\end{align*}
\]

\[ (2.31) \]

- \( \rho_{\text{act}} > \rho_b \) is not allowed by code, in this case we have an extra unknown \( f_s \).

We now have one more unknown \( f_s \), and we will need an additional equation (from strain diagram).

\[
\begin{align*}
c &= \frac{A_s f_y}{0.85 f'_c} \\
\bar{c} &= \frac{0.003}{0.003 + \varepsilon_s} \\
M_d &= \phi A_s f_s (d - \frac{a}{2}) \quad \Sigma M = 0
\end{align*}
\]

\[ (2.32) \]

We can solve by iteration, or substitution and solution of a quadratic equation.

2.3.4 Design

We consider two cases:

I. \( b, d \) and \( A_s \), unknown; \( M_d \) known; Since design failure is triggered by \( f_s = f_y \)

\[
\begin{align*}
\Sigma F_x &= 0 \\
\rho &= \frac{A_s f_y}{0.85 f'_c} \\
\begin{cases}
a = \frac{\rho f_y}{0.85 f'_c} \\
M_d = A_s f_y \left( d - \frac{a}{2} \right)
\end{cases}
\end{align*}
\]

\[ (2.33-a) \]

where \( \rho \) is specified by the designer; or

\[
R = \rho f_y \left( 1 - 0.59 \frac{f_y}{f'_c} \right)
\]

\[ (2.34) \]
2.4 Practical Design Considerations

2.4.2 Beam Sizes, Bar Spacing, Concrete Cover

Beam sizes should be dimensioned as
1. Use whole inches for overall dimensions, except for slabs use $\frac{1}{2}$ inch increment.
2. Ideally, the overall depth to width ratio should be between 1.5 to 2.0 (most economical).
3. For T beams, flange thickness should be about 20% of overall depth.

Reinforcing bars
1. Minimum spacing between bars, and minimum covers are needed to
   (a) Prevent Honeycombing of concrete (air pockets)
   (b) Concrete (usually up to 3/4 in MSA) must pass through the reinforcement
   (c) Protect reinforcement against corrosion and fire
2. Use at least 2 bars for flexural reinforcement
3. Use bars #11 or smaller for beams.
4. Use no more than two bar sizes and no more than 2 standard sizes apart (i.e. #7 and #9 acceptable; #7 and #8 or #7 and #10 not).
5. Use no more than 5 or 6 bars in one layer.
6. Place longest bars in the layer nearest to face of beam.
7. Clear distance between parallel bars not less that $d_b$ (to avoid splitting cracks) nor 1 in. (to allow concrete to pass through).
8. Clear distance between longitudinal bars in columns not less that $1.5d_b$ or 1.5 in.
9. Minimum cover of 1.5 in.
10. Summaries in Fig. 2.7 and Table 2.1, 2.2.

2.4.3 Design Aids

Basic equations developed in this section can be easily graphed.

Review Given $b \ d$ and known steel ratio $\rho$ and material strength, $\phi M_n$ can be readily obtained from $\phi M_n = \phi Rbd^2$

Design in this case
1. Set $M_d = \phi Rbd^2$
2. From tabulated values, select $\rho_{\text{max}}$ and $\rho_{\text{min}}$ often $0.5\rho_b$ is a good economical choice.
3. Select $R$ from tabulated values of $R$ in terms of $f_y$, $f'_c$ and $\rho$. Solve for $bd^2$.
4. Select $b$ and $d$ to meet requirements. Usually depth is about 2 to 3 times the width.
5. Using tabulated values select the size and number of bars giving preference to larger bar sizes to reduce placement cost (careful about crack width!).
6. Check from tables that the selected beam width will provide room for the bars chosen with adequate cover and spacing.
2.5 USD Examples

Example 2-5: Ultimate Strength; Review

Determine the ultimate moment capacity of example 2.1 $f'_c = 4,000$ psi; $f'_t = 475$ psi; $f_y = 60,000$ psi; $A_s = 2.35\text{ in}^2$

Solution:

\[ \rho_{act} = \frac{A_s}{bd} = \frac{2.35}{(10)(23)} = 0.0102 \]  \hfill (2.39-a)

\[ \rho_b = 0.85\beta_1 \frac{f'_c}{f_y} \frac{87}{87 + f_y} = (0.85)(0.85) \frac{4}{60} \frac{87}{87 + 60} = 0.0285 > \rho_{act} \sqrt{a} \]  \hfill (2.39-b)

\[ a = \frac{A_s f_y}{0.85 f'_t b} = \frac{(2.35)(60)}{(0.85)(4)(10)} = 4.15 \text{ in} \]  \hfill (2.39-c)

\[ M_n = A_s f_y \left( d - \frac{a}{2} \right) = (2.35)(60) \left( 23 - \frac{4.15}{2} \right) = 2,950 \text{ k.in} \]  \hfill (2.39-d)

\[ M_d = \phi M_n = 0.9(2,950) = 2,660 \text{ k.in} \]  \hfill (2.39-e)

Note:
Example 2-7: Ultimate Strength; Design II

Design a R/C beam for \( b = 11.5 \) in; \( d = 20 \) in; \( f'_c = 3 \) ksi; \( f_y = 40 \) ksi; \( M_d = 1,600 \) k.in

Solution:

Assume \( a = \frac{d}{5} = \frac{20}{5} = 4 \) in

\[
A_s = \frac{M_d}{\phi f_y (d - \frac{a}{3})} = \frac{(1,600)}{(0.9)(40)(20 - \frac{4}{3})} = 2.47 \text{ in}^2 \quad (2.42)
\]

Check assumption,

\[
a = \frac{A_s f_y}{(0.85)f'_c b} = \frac{(2.47)(40)}{(0.85)(3)(11.5)} = 3.38 \text{ in} \quad (2.43)
\]

Thus take \( a = 3.3 \) in.

\[
A_s = \frac{(1,600)}{(0.9)(40)(20 - \frac{3.3}{2})} = 2.42 \text{ in}^2 \quad (2.44-a)
\]

\[
\Rightarrow a = \frac{(2.42)(40)}{(0.85)(3)(11.5)} = 3.3 \text{ in} \quad (2.44-b)
\]

\[
\rho_{act} = \frac{2.42}{(11.5)(20)} = 0.011 \quad (2.44-c)
\]

\[
\rho_b = \frac{(0.85)(0.85)}{40} \frac{87}{87 + 40} = 0.037 \quad (2.44-d)
\]

\[
\rho_{max} = 0.75 \rho_b = 0.0278 > \rho_{act} \sqrt{ } \quad (2.44-e)
\]

Example 2-8: Exact Analysis

As an Engineer questioning the validity of the ACI equation for the ultimate flexural capacity of R/C beams, you determined experimentally the following stress strain curve for concrete:

\[
\sigma = \frac{2 f'_c}{\varepsilon_{max}} \frac{\varepsilon}{1 + \left( \frac{\varepsilon}{\varepsilon_{max}} \right)^2} \quad (2.45)
\]

where \( f'_c \) corresponds to \( \varepsilon_{max} \).

1. Determine the exact balanced steel ratio for a R/C beam with \( b = 10'' \), \( d = 23'' \), \( f'_c = 4,000 \) psi, \( f_y = 60 \) ksi, \( \varepsilon_{max} = 0.003 \).

(a) Determine the equation for the exact stress distribution on the section.

(b) Determine the total compressive force \( C \), and its location, in terms of the location of the neutral axis \( c \).
Chapter 3

SHEAR

3.1 Introduction

1. Beams are subjected to both flexural and shear stresses. Resulting principal stresses (or stress trajectory) are shown in Fig. 3.1.

![Diagram of principal stresses in beam](image)

Figure 3.1: Principal Stresses in Beam

2. Due to flexure, vertical flexural cracks develop from the bottom fibers.

3. As a result of the tensile principal stresses, two types of shear cracks may develop, Fig. 3.2:

![Diagram of shear cracks](image)

Figure 3.2: Types of Shear Cracks

**Web shear cracks:** Large V, small M. They initiate in the web & spread up & down at $\approx 45^\circ$. 
3. Compute the principal stresses
4. Equate principal tensile stress to the tensile strength

Using a semi-analytical approach
1. Assume that $f_c$ is directly proportional to steel stress

$$
\begin{align*}
    f_c &= \alpha \frac{f_s}{E_s} \\
    M_n &= A_s f_s j d \Rightarrow f_s = \frac{M_n}{A_s j d} \quad \left\{ \begin{array}{l}
    f_c = \alpha M_n \\
    \rho = \frac{A_s}{n j d}
\end{array} \right. \\
    f_c &= \alpha \frac{M_n}{n j p j b d^2} = F_1 \frac{M_n}{j p b d^2}
\end{align*}
$$

(3.1)

2. Shear stress

$$
    v_n = F_2 \frac{V_n}{b d}
$$

(3.2)

3. From Mohr’s circle, the tensile principal stress is

$$
    f_1 = \frac{f_c}{2} + \sqrt{\left(\frac{f_c}{2}\right)^2 + v_n^2}
$$

(3.3)

4. Set $f_1$ equal to the tensile strength

$$
    \begin{align*}
    f_1 &= f'_1 
    V_n &= f'_1 \frac{V_n}{b d}
    V_n &= f'_1 \frac{V_n}{f'_1 b d}
    &= \frac{f'_1}{f'_1 b d}
\end{align*}
$$

(3.4-a, 3.4-b, 3.4-c)

Combining Eq. 3.1, 3.2, and 3.3

$$
\begin{align*}
    \frac{V_n}{b d} &= \frac{f'_1}{2} \left[ \frac{F_1 E_c M_n}{E_s p V_n d} + \left( \frac{F_1 E_c M_n}{E_s p V_n d} \right)^2 \right]^{1/2}
\end{align*}
$$

(3.5)