ABSTRACT

Bifurcation of an elastoplastic dynamic saturated porous medium has been shown to be related to the underlying solid skeleton drained behavior only, in the small strain regime [1]. Such analysis is extended to a simple non-associative Drucker-Prager cap plasticity model at finite strain implemented in an implicit dynamic saturated biphasic (solid and fluid) three-dimensional finite element [2]. Formulation of the plasticity model assumes a multiplicative decomposition of the deformation gradient into elastic and plastic parts, and the model equations are expressed in the intermediate configuration.

INTRODUCTION

There are a number of engineering problems of interest that involve saturated porous media, for which three-dimensional (3D), finite strain, dynamic finite element (FE) analysis of inelastic deformation, effective stress, pore fluid pressure, and bifurcation is important. Such problems include building foundations resting on saturated soils subjected to earthquake loading (in various directions), embankment dams subjected to earthquake loading, vertebral disks subjected to dynamic loads during running, jumping, or impact, etc.

This paper presents preliminaries of a 3D, finite strain, dynamic biphasic mixture elastoplasticity model formulated for FE implementation, with details left to other papers [2,3]. The main contribution of this paper is to present the bifurcation analysis of strain localization (continuous bifurcation, weak discontinuity [4]) for an elastoplastic, saturated biphasic mixture, finite strain simple geomaterial. Previous analyses by Loret and Prevost [1], Rudnicki [5], Callari and Armero [6] have shown that the underlying solid skeleton drained material behavior governs bifurcation, although can be delayed for low permeability materials as the effective stress is influenced by the evolving pore fluid pressure.

We use lower case letters for variables and coordinates in the current configura-

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tion (such as differential volume dv), capital letters for the reference configuration (such as differential volume dV), and capital letters with an overbar for the intermediate configuration (such as differential volume $d\bar{V}$) when assuming a multiplicative decomposition of the deformation gradient into elastic and plastic parts. Bold-face letters denote matrices, tensors and vectors; the symbol "$\cdot$" denotes an inner product of two vectors (e.g., $\boldsymbol{a}\cdot\boldsymbol{b} = a_i b^i$), or a single contraction of adjacent indices of two tensors (e.g., $\boldsymbol{c}\cdot\boldsymbol{d} = c_{ij}d^j_k$); the symbol ":" denotes an inner product of two second order tensors (e.g., $\boldsymbol{c}:\boldsymbol{d} = c_{ij}d^{ij}$), or a double contraction of adjacent indices of tensors of rank two and higher (e.g., $\boldsymbol{D}:\boldsymbol{C} = D_{IJKL}C^{KL}$); the symbol "$\otimes$" denotes dyadic product of two tensors (e.g., $\boldsymbol{c}\otimes\boldsymbol{d} = c_{ij}d_{kl}$). "grad" and "div" represent the gradient and divergence operators with respect to the current configuration, respectively; "GRAD" and "DIV" represent the gradient and divergence operators with respect to the reference configuration, respectively. Super-script and sub-script indices distinguish between contravariant and covariant tensor components, respectively. We will assume Cartesian coordinates for the reference and current configurations.

VOLUME FRACTION AND KINEMATICS

The volume fraction n^α for constituent α is defined as

$$n^\alpha \stackrel{\text{def}}{=} dv_\alpha/dv\ ,\ \sum_\alpha n^\alpha = 1\ ,\ \sum_\alpha dv_\alpha = dv \tag{1}$$

where α =s,f (solid and fluid constituents), differential mixture volume $dv \subset \mathcal{B}$, $\mathcal{B} = \mathcal{B}^{\mathrm{s}}$ (the mixture control space is that of the solid phase; i.e., we follow the solid phase motion), and dv_α is the differential volume of constituent α. Let $\mathcal{B}_0$ and $\mathcal{B}_0^{\mathrm{f}}$ denote the reference configurations of the solid and fluid phases, respectively. We drop the s designation because the solid phase motion represents that of the mixture. Plastic deformation for the solid skeleton is introduced through a multiplicative decomposition of the solid skeleton deformation gradient as $\boldsymbol{F} = \boldsymbol{F}^e\boldsymbol{F}^p$ [7]. Clayton et al. [8] derived deformation measures associated with the multiplicative decomposition of $\boldsymbol{F}$ and the existence of a non-Euclidean intermediate configuration $\bar{\mathcal{B}}$. It was shown that the covariant components of certain tensors in $\bar{\mathcal{B}}$ contain the covariant metric coefficients $\bar{G}_{\bar{K}\bar{L}}$. The multiplicative decomposition is written as

$$\boldsymbol{F} = \boldsymbol{F}^e\cdot\boldsymbol{F}^p = F^{ek}_{\cdot\bar{K}}F^{p\bar{K}}_{\cdot K}\boldsymbol{g}_k\otimes\boldsymbol{G}^K \tag{2}$$

where $\boldsymbol{G}^K$ are the contravariant basis vectors in $\mathcal{B}_0$, and $\boldsymbol{g}_k$ the covariant basis vectors in $\mathcal{B}$. One choice for the covariant metric coefficients on $\bar{\mathcal{B}}$ are [8] $\bar{G}_{\bar{K}\bar{L}} \stackrel{\text{def}}{=} \bar{C}^e_{\bar{K}\bar{L}} = F^{ek}_{\cdot\bar{K}}g_{kl}F^{el}_{\cdot\bar{L}}$.

THERMODYNAMICS

The first and second laws of thermodynamics, upon introducing the Helmholtz free energy function ψ^α for constituent α, combine to give the Clausius-Duhem inequality for the biphasic mixture in the current configuration $\mathcal{B}$ as [9] (eq.5.7)

$$-\rho^{\mathrm{s}}\frac{D\psi^{\mathrm{s}}}{Dt}-\rho^{\mathrm{f}}\frac{D^{\mathrm{f}}\psi^{\mathrm{f}}}{Dt}+\boldsymbol{\sigma}^{\mathrm{s}}:\boldsymbol{\ell}+\boldsymbol{\sigma}^{\mathrm{f}}:\boldsymbol{\ell}_{\mathrm{f}}-\boldsymbol{h}^{\mathrm{f}}\cdot\tilde{\boldsymbol{v}}_{\mathrm{f}}\geq 0 \tag{3}$$

where ρ^{α} are the partial mass densities, $D^{\alpha}(\bullet)/Dt$ are the material time derivatives with respect to their constituent motion (no superscript implies solid motion), $\boldsymbol{\sigma}^{\alpha}$ are the constituent Cauchy stresses, $\boldsymbol{\ell}_{\alpha}$ the velocity gradients, $\boldsymbol{h}^{\mathrm{f}}$ the interconstituent force on the fluid by the solid (as the fluid flows through the pores), and $\tilde{\boldsymbol{v}}_{\mathrm{f}}$ the relative fluid velocity. Sparing details presented in [2], the reduced dissipation inequality results as

$$-\int_{\mathcal{B}}\left(\mathrm{grad}(p_{\mathrm{f}})+\rho^{\mathrm{f}R}(\boldsymbol{a}_{\mathrm{f}}-\boldsymbol{g})\right)\cdot\tilde{\boldsymbol{v}}^{\mathrm{f}}dv+\int_{\bar{\mathcal{B}}}\left(\bar{\boldsymbol{S}}':(\bar{\boldsymbol{C}}^{e}\cdot\bar{\boldsymbol{L}}^{p})-\bar{\boldsymbol{Q}}\cdot\dot{\bar{\boldsymbol{Z}}}\right)d\bar{V}\geq 0 \tag{4}$$

where p_{f} is the pore fluid pressure, $\rho^{\mathrm{f}R}$ the real fluid mass density, $\boldsymbol{a}_{\mathrm{f}}$ the fluid acceleration, $\tilde{\boldsymbol{v}}^{\mathrm{f}}=n^{\mathrm{f}}\tilde{\boldsymbol{v}}_{\mathrm{f}}$ the Eulerian Darcy relative fluid velocity, $\bar{\boldsymbol{S}}'$ the Second Piola-Kirchhoff effective stress in the intermediate configuration, $\bar{\boldsymbol{L}}^{p}$ the plastic velocity gradient in the intermediate configuration, $\bar{\boldsymbol{Q}}$ a vector of stress-like internal state variables (ISVs) and $\bar{\boldsymbol{Z}}$ a vector of strain-like ISVs in $\bar{\mathcal{B}}$. The conjugate terms will provide the variables for the Darcy relative fluid velocity, plastic deformation rate, and internal state variable evolution equations. Constitutive forms for the pore fluid pressure and effective stress result as [2]

$$p_{\mathrm{f}}=(\rho^{\mathrm{f}R})^{2}\frac{\partial\psi^{\mathrm{f}}}{\partial\rho^{\mathrm{f}R}}\,,\quad \bar{\boldsymbol{S}}'=2\bar{\rho}^{\mathrm{s}}\frac{\partial\bar{\psi}^{\mathrm{s}}}{\partial\bar{\boldsymbol{C}}^{e}} \tag{5}$$

CONSTITUTIVE EQUATIONS

Constitutive equations need to be defined for the Helmholtz free energy functions for solid and fluid phases, including plastic evolution equations for the solid phase skeleton. **Fluid phase**: Assuming barotropic flow and constant fluid bulk modulus K_{f}, the Helmholtz free energy function for the fluid constituent leads to [2] $p_{\mathrm{f}}=K_{\mathrm{f}}\ln\rho^{\mathrm{f}R}$. A constitutive equation for the Eulerian Darcy relative fluid velocity $\tilde{\boldsymbol{v}}^{\mathrm{f}}$ that is proportional to the first term in (4) ensures non-negative dissipation contribution by the fluid constituent term in the dissipation inequality. Such a constitutive equation was proposed by Coussy [10] (Sect.3.3.1) where porosity-dependent isotropic hydraulic conductivity $k(n^{\mathrm{f}})$ is assumed. Furthermore, we will assume the relative fluid acceleration vector is approximately zero (appropriate for intermediate to long period motions like earthquakes and athletic activities), such that $\boldsymbol{a}_{\mathrm{f}}=\boldsymbol{a}$. More details can be found in [9,10]. **Solid phase:** For the solid skeleton, we additively separate the Helmholtz free energy function into elastic and plastic parts as

$$\bar{\psi}^{\mathrm{s}}(\bar{\boldsymbol{C}}^{e},\bar{\boldsymbol{Z}})=\bar{\psi}^{\mathrm{s},e}(\bar{\boldsymbol{C}}^{e})+\bar{\psi}^{\mathrm{s},p}(\bar{\boldsymbol{Z}}) \tag{6}$$

We assume neo-Hookean compressible elasticity [11], such that the constitutive equation for effective Second Piola Kirchhoff stress in $\bar{\mathcal{B}}$ is

$$\bar{\boldsymbol{S}}'=\mu\bar{\boldsymbol{G}}+(\lambda\ln J^{e}-\mu)\bar{\boldsymbol{C}}^{e-1} \tag{7}$$

where λ and μ are the Lamé constants, $J^e = \det \boldsymbol{F}^e$. For the plastic part of the Helmholtz free energy function, we define it such that [2] $\bar{\boldsymbol{Q}} = \bar{\boldsymbol{H}} \cdot \bar{\boldsymbol{Z}}$, where $\bar{\boldsymbol{H}}$ is a matrix of linear hardening/softening parameters. Now, it remains to define evolution equations for the plastic velocity gradient $\bar{\boldsymbol{L}}^p$ and strain-like ISV $\dot{\bar{\boldsymbol{Z}}}$ in the solid constituent contribution to the reduced dissipation inequality. With the covariant metric coefficients defined in $\bar{\mathcal{B}}$, the covariant coefficients of the plastic velocity gradient are

$$\bar{L}^p_{\bar{K}\bar{L}} = \bar{G}_{\bar{K}\bar{A}} \bar{L}^{p\bar{A}}_{\cdot\bar{L}} = \bar{C}^e_{\bar{K}\bar{A}} \bar{L}^{p\bar{A}}_{\cdot\bar{L}} \ , \ \bar{L}^{p\bar{A}}_{\cdot\bar{L}} = \dot{F}^{p\bar{A}}_{\cdot K} F^{p-1K}_{\cdot\bar{L}} \tag{8}$$

We assume the plastic spin is zero $\bar{W}^p_{\bar{K}\bar{L}} = 0$, such that the plastic deformation rate is $\bar{D}^p_{\bar{K}\bar{L}} = \bar{L}^p_{\bar{K}\bar{L}}$. We define the plastic deformation rate as

$$\bar{D}^p_{\bar{K}\bar{L}} = \dot{\bar{\gamma}} \frac{\partial \bar{\mathcal{G}}}{\partial \bar{S}'^{\bar{K}\bar{L}}} \tag{9}$$

where $\dot{\bar{\gamma}}$ is the plastic multiplier, and $\bar{\mathcal{G}}(\bar{\boldsymbol{S}}', \bar{\boldsymbol{Q}})$ is a plastic potential function. Also $\dot{\bar{\boldsymbol{Z}}} = \dot{\bar{\gamma}} \bar{\boldsymbol{h}}^Z$. Details of the plasticity model are presented in [2].

BIFURCATION ANALYSIS

A finite element bifurcation analysis typically involves the following steps: (1) a statement of the variational form of the mixture balance of linear momentum equations for kinematic discontinuities to identify jump terms, (2) definition of kinematic discontinuities, (3) formulation of bifurcation criteria depending on the constitutive model and kinematics, (4) implementation of such criteria into a finite element code, and (5) analysis of numerical examples for determining onset of localized deformation for certain material parameters and boundary conditions. Steps (1-3) will be treated here. Steps (4,5) are left for future work.

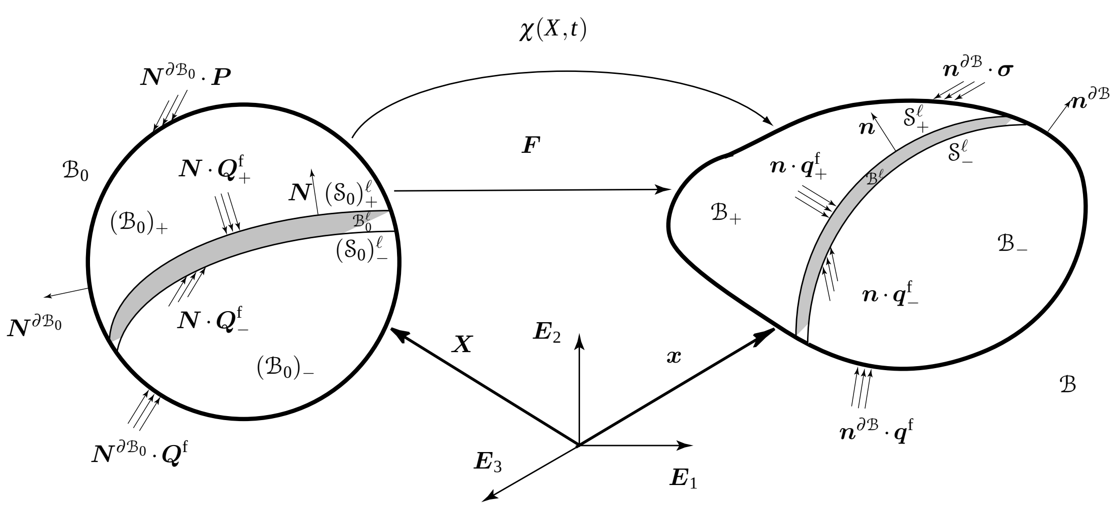

Figure 1. Biphasic porous body with weak discontinuity. Reference configuration $\mathcal{B}_0$ mapped to current configuration $\mathcal{B}$ with motion $\chi(\boldsymbol{X},t)$. Essential BCs are not shown.

(1) Variational form of the mixture balance of linear momentum: Referring to Figure 1, the variational form of the mixture balance of linear momentum with weak discontinuity is

$$
\begin{aligned}
& \int_{\mathcal{B}_0} \boldsymbol{w} \cdot \left(\rho_0 \frac{\partial \boldsymbol{V}}{\partial t} + \rho_0^{\mathrm{f}} \frac{\partial \tilde{\boldsymbol{V}}^{\mathrm{f}}}{\partial t} \right) dV + \int_{\mathcal{B}_0} \mathrm{GRAD}\boldsymbol{w} : \boldsymbol{P} dV \\
& + \int_{\mathcal{B}_0} \mathrm{GRAD}\boldsymbol{w} : \left((\boldsymbol{V} + \tilde{\boldsymbol{V}}^{\mathrm{f}}) \otimes \boldsymbol{Q}^{\mathrm{f}} \right) dV + \int_{\mathcal{B}_0} \boldsymbol{w} \cdot \left((\boldsymbol{V} + \tilde{\boldsymbol{V}}^{\mathrm{f}}) \mathrm{DIV}\boldsymbol{Q}^{\mathrm{f}} \right) dV \\
= \; & \int_{\mathcal{B}_0} \boldsymbol{w} \cdot (\rho_0 \boldsymbol{g}) dV + \int_{(\partial\mathcal{B}_0)_t} \boldsymbol{w} \cdot \boldsymbol{T}_t dA + \int_{(\partial\mathcal{B}_0)_q} \boldsymbol{w} \cdot \left((\boldsymbol{V} + \tilde{\boldsymbol{V}}^{\mathrm{f}}) \otimes \boldsymbol{Q}^{\mathrm{f}} \right) \cdot \boldsymbol{N}^{\partial\mathcal{B}_0} dA \\
& + \int_{(\mathcal{S}_0)_+^{\ell}} \boldsymbol{w} \cdot ([\![\boldsymbol{P}]\!] \cdot \boldsymbol{N}) \, dA + \int_{(\mathcal{S}_0)_-^{\ell}} \boldsymbol{w} \cdot ([\![\boldsymbol{P}]\!] \cdot \boldsymbol{N}) \, dA
\end{aligned} \tag{10}
$$

where $\boldsymbol{w}$ is the weighting function, ρ_0 is the total mixture reference mass density, ρ_0^{f} the partial fluid reference mass density, $\boldsymbol{V}$ the solid material velocity vector, $\tilde{\boldsymbol{V}}^{\mathrm{f}}$ the relative fluid material velocity vector, $\boldsymbol{P}$ the total mixture first Piola-Kirchhoff stress, $\boldsymbol{Q}^{\mathrm{f}}$ the interstitial material fluid flow vector, $\boldsymbol{g}$ the gravitational acceleration, $\boldsymbol{T}_t$ the traction in the current configuration with respect to the reference area, $[\![\boldsymbol{P}]\!]$ the jump in $\boldsymbol{P}$ across the discontinuity surface, $\boldsymbol{N}$ the unit normal to the planar weak discontinuity (at a material point), and $\boldsymbol{N}^{\partial\mathcal{B}_0}$ the normal to the boundary. Note that the flux $\boldsymbol{Q}^{\mathrm{f}} \cdot \boldsymbol{N}$ is continuous across the band (this is not the case for strong discontinuity). For traction across the band to be continuous, the following condition must hold across the discontinuity surfaces $[\![\boldsymbol{P}]\!] \cdot \boldsymbol{N} = \boldsymbol{0}$. This leads to the bifurcation condition, where the total stress is composed of the partial stresses $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{s}} + \boldsymbol{P}^{\mathrm{f}}$, and thus constitutive equations for solid and fluid constituents must be accounted for.

(2) Kinematics: Assuming a planar band of weak discontinuity with thickness ℓ_0 parameterized by η ($0 \le \eta \le \ell_0$), the displacement becomes

$$
\boldsymbol{u}(\boldsymbol{X},t) = \tilde{\boldsymbol{u}}(\boldsymbol{X},t) + \eta \, [\![\boldsymbol{u}(\boldsymbol{X},t)]\!] / \ell_0 \tag{11}
$$

where $\tilde{\boldsymbol{u}}(\boldsymbol{X},t)$ is the compatible displacement, $[\![\boldsymbol{u}(\boldsymbol{X},t)]\!]$ the jump in displacement across the discontinuity $\mathcal{B}_0^{\ell}$, and $\boldsymbol{N}$ the unit normal to $(\mathcal{S}_0)_+^{\ell}$ and $(\mathcal{S}_0)_-^{\ell}$. This displacement leads to a deformation gradient $\boldsymbol{F}$ of the form

$$
\boldsymbol{F} = \boldsymbol{1} + \mathrm{GRAD}\tilde{\boldsymbol{u}} + \mathrm{GRAD}\left([\![\boldsymbol{u}]\!]\right) \eta / \ell_0 + \left([\![\boldsymbol{u}]\!] \otimes \boldsymbol{N}\right) / \ell_0 \tag{12}
$$

(3) Bifurcation criterion: We limit the analysis to weak discontinuity kinematics, and express the traction continuity condition in rate form as $[[\dot{\boldsymbol{P}}]] \cdot \boldsymbol{N} = \boldsymbol{0}$ or $\frac{1}{J}(\dot{\boldsymbol{P}}^b - \dot{\boldsymbol{P}}^o) \cdot \boldsymbol{F}^T \cdot \boldsymbol{n} = \boldsymbol{0}$ where "b" is within the band and "o" just outside the band. We assume the effective stress, pore fluid pressure, and elastic deformation are continuous across $(\mathcal{S}_0)_+^{\ell}$ and $(\mathcal{S}_0)_-^{\ell}$ (i.e., $\boldsymbol{\sigma}'$, p_{f}, $\boldsymbol{F}^e$ continuous). For continuous bifurcation (plastic loading within and outside the band at bifurcation), the plastic multiplier is $\bar{\dot{\gamma}} = \bar{\dot{\gamma}}^{\mathrm{reg}} + \bar{\dot{\gamma}}_{\ell_0} / \ell_0$. For a planar band, $[\![\dot{\boldsymbol{u}}]\!] = \dot{\zeta} \boldsymbol{m}$, where $\boldsymbol{m}$ is the direction of jump displacement and $\dot{\zeta}$ its magnitude in $\mathcal{B}$. Skipping many details, a bifurcation condition results as $\boldsymbol{A} \cdot \boldsymbol{m} = \boldsymbol{0}$, det$\boldsymbol{A}$=0 for $\boldsymbol{m} \neq \boldsymbol{0}$, where $\boldsymbol{A}(\boldsymbol{\sigma}', \boldsymbol{F}^e, \bar{\boldsymbol{Q}})$ is the acoustic tensor and not a function of p_{f} (although the effective stress is influenced by the evolution of pore fluid pressure).

SUMMARY

The paper presented a thermodynamically consistent finite strain elastoplastic biphasic mixture model including inertia terms [2]. Bifurcation analysis assuming continuous bifurcation, weak discontinuity, led to a condition dependent on effective stress, elastic deformation, and stress-like ISVs, but no explicit dependence on the pore fluid pressure. The effective stress will depend on the evolution of the pore fluid pressure during solution of the initial boundary value problem, even though the bifurcation condition relies only on analysis of the solid skeleton constitutive equations. Numerical examples demonstrating the bifurcation condition will be presented in a future paper.

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